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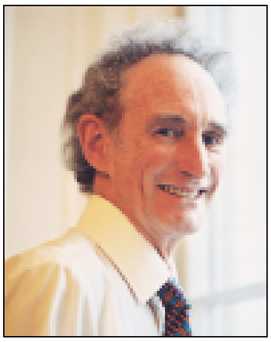
Teaching and learning geometry 11-19

Report of a Royal Society / Joint Mathematical Council working group

Foreword

**Lord May of Oxford, President of the Royal Society and
Professor Celia Hoyles, Chair of the Joint Mathematical Council**



Lord May of Oxford AC PRS

Professor Celia Hoyles

Following the 1997 publication of the Royal Society and Joint Mathematical Council report on the teaching and learning of algebra, representatives from all quarters of the mathematics community began to call for a similar study into the teaching of geometry. In response to these calls, the Royal Society and JMC arranged a seminar in October 1999 to enable discussion about the place of geometry in the National Curriculum and post-16 education.

A wide range of opinion was voiced at the seminar, but all those present agreed upon three things: that geometry was of vital importance to the mathematical education of all young people; that the geometry component of 11-19 mathematics could be improved significantly from its current position; and that there would thus be value in establishing a working group under the auspices of the JMC and Royal Society to examine the issues in detail.

This report is the result of the working group's discussions. It argues for considerable changes in the way that geometry is taught 11-16 (including a significant commitment to teachers' continuing professional development) and for a fundamental review of the structure of post-16 qualifications in mathematics.

On behalf of the Royal Society and JMC, we wish to express our thanks to Professor Adrian Oldknow and the members of the working group for the substantial effort they have put into this study.

We commend this report to policy makers throughout the education system.

Cover picture: The Great Court of the British Museum showing the southern portico below the glass and steel roof – an elegant combination of classical and modern geometry. (Photograph by Dudley Hubbard. © The British

Teaching and learning geometry 11-19

Contents

Contents (continued)

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Preparation of this report

This report has been endorsed by the Council of the Royal Society and the JMC. It has been prepared by the Royal Society / Joint Mathematical Council working group on the teaching and learning of geometry.

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The working group wishes to record its thanks to Professor Geoffrey Howson who made valuable contributions to the group's work and also to the following individuals who acted as 'readers': Dr Bob Burn; Tandi Clausen-May; David Fielker; Dr Tony Gardiner; Professor Celia Hoyles; Mike Ollerton; and Dr Leo Rogers (who also provided advice relating to Appendix 5).

Chairman's preface

Adrian Oldknow

"About binomial theorem I'm teeming with a lot of news,
With many cheerful facts about the square of the hypotenuse"
(WS Gilbert, *The Pirates of Penzance*)

The mathematical content for pupils following the National Curriculum in secondary schools in England is described under the headings of: *Number and algebra*; *Shape, space and measures* and *Handling data*. However the term 'numeracy' has become increasingly used in place of mathematics in relation to school education. This is an unfortunate practice since it downplays two areas, algebra and geometry, which are of major importance in school mathematics. The teaching of each of these aspects of mathematics has now been the subject of commissioned reports from the Royal Society and the Joint Mathematical Council of the United Kingdom, and the Qualifications and Curriculum Authority is currently engaged in a three year project on developing the teaching of both algebra and geometry.

A past President of the Royal Society, Sir Michael Atiyah, provided some succinct background to the development of algebra and geometry in a lecture given in Toronto in June 2000:

> *I want to talk now about a dichotomy in mathematics, which has been with us all the time, oscillating backwards and forwards... I refer to the dichotomy between geometry and algebra. Geometry and algebra are the two formal pillars of mathematics; they both are very ancient. Geometry goes back to the Greeks and before; algebra goes back to the Arabs and the Indians, so they have both been fundamental to mathematics, but they have had an uneasy relationship.*
> *Let me start with the history of the subject. Euclidean geometry is the prime example of a mathematical theory and it was firmly geometrical, until the introduction by Descartes of algebraic coordinates, in what we now call the Cartesian plane. That was an attempt to reduce geometrical thinking to algebraic manipulation.*
> (Reprinted in *Mathematics Today*, **37**(2), April 2001 46-53.)

At school level, algebra can seem quite abstract and cerebral. In Fitzgerald's studies for the Cockcroft committee it was algebra which was most frequently cited as the part of mathematics where adults remembered losing touch with mathematics. On the other hand, there are clear links in geometry to the world of our senses and experience. For example, we can easily perceive when objects are parallel, or perpendicular, or symmetrical - such as recognising when a minute adjustment is needed to the way a picture hangs. Sir Michael offers the following comments on our capacity to perceive, and its relationship with geometry:

> *Our brains have been constructed in such a way that they are extremely concerned with vision. Vision, I understand from friends who work in neurophysiology, uses up something like 80 or 90 percent of the cortex of the brain...*
> *Understanding, and making sense of, the world that we see is a very important part of our evolution. Therefore spatial intuition or spatial perception is an enormously powerful tool and that is why geometry is actually such a powerful part of mathematics - not only for things that are obviously geometrical, but even for things that are not. We try to put them into geometrical form because that enables us to use our intuition. Our intuition is our most powerful tool... I think it is very fundamental that the human mind has evolved with this enormous capacity to absorb a vast amount of information, by instantaneous visual action, and mathematics takes that and perfects it.*

Geometry is of far reaching importance beyond the worlds of professional mathematicians and of mathematics teaching. Geometry is frequently used to model what we call the 'real world' and has many applications in solving practical problems. (It is interesting to note that the French term for a surveyor is '*un expert géomètre*'.) Geometry is making contributions to many important scientific developments such as the Human Genome Project, Buckminster-Fullerene research, and whole-body tomography. Through media such as film, television and computer games we encounter computer generated geometric images of great complexity, and children and adults alike derive pleasure from creating designs and patterns exhibiting geometric forms.

So geometry is an important subject, with wide applications and a long history. It deals with matters we find attractive and for which we have a strong visual capacity. On the surface, then, it would appear that geometry should be one of the easiest branches of mathematics to teach. But this is not the case - neither in England nor in much of the developed world. This Royal Society / JMC study set out to identify why this is so.

Geometry is one of the oldest branches of mathematics - itself one of the oldest of mankind's intellectual studies. No wonder, then, that it suffers from an embarrassment of riches in terms of theories, results, techniques and applications. Many of these are well within the grasp of most, if not all, students in 11-19 education. We might refer to this, not unwelcome, problem as one of **abundance.** Clearly, then, choices have to be made on what material to include in the curriculum. At one extreme there is a danger of choosing eclectically from this abundance in a way that leads to the teaching of a lot of apparently unconnected 'bits'. At the other extreme there is a danger of developing a tightly organised body of knowledge which addresses only a very small part of geometry. Our challenge has been to combine breadth with both educational and mathematical coherence - a problem we refer to as **coherence**.

One of the less obvious difficulties in teaching geometry lies in the abstractions we make – we illustrate points and line segments through drawings and diagrams and yet neither object can be visible, except in our 'mind's eye'. We do not often choose to discuss such a difficult issue! Frequently however, teachers will draw rapid sketches purporting to represent objects in their own imagination which may actually not be recognised as such by their pupils.

The geometry of the ancient Greeks, as recorded by Euclid, was far more than a summary of known facts - it was an organised body of knowledge starting with a number of definitions and assumptions (axioms) which used logical deduction to establish a series of results in the form of theorems together with proofs. It is through the teaching of geometry that most pupils still encounter at least one theorem, that of Pythagoras, together with one or more proofs, and maybe some applications. My non-scientific guess is that most adults will remember the name Pythagoras, and probably that his theorem has to do with right angled triangles and words like 'hypotenuse', but that it would be extraordinary if they could remember a proof of the theorem. The role of proof, and the range of pupils for whom it is relevant, remains a major issue in geometry teaching. Despite the long tradition for the inclusion of geometrical proof in school curricula there is little evidence that we have developed effective methods for its teaching. Nevertheless, the working group supports the inclusion of proof in school geometry both because of its central role in mathematics, and as a contribution to developing more general skills of argument and criticism.

In order to address the issue of coherence the working group has followed on from a previous review of the geometry curriculum [Wynne Willson, 1977] and formulated a set of objectives for the teaching of geometry in the 21st Century. Against these objectives we have concluded that the geometrical content of the National Curriculum does provide a reasonable basis for the 11-16 curriculum, but needs strengthening in two main areas. These concern work in 3-dimensions, and in the educational application of Information and Communications Technology (ICT). Leaving the geometrical content relatively unchanged for now will allow scope for dealing with the issue which the working group has identified as by far the most important one for 11-16 geometry. That is to ensure that teachers have the knowledge, understanding, skills and resources to teach geometry in a way which genuinely captures pupils' interest and imagination, while developing their thinking and reasoning skills, their powers of visualisation, their ability to apply and model, and their understanding.

The 11-16 geometry curriculum in England continues to concentrate on techniques for working in 2 dimensions, such as the plane geometry derived from Euclid, together with elements of transformation, vector and coordinate geometry. Yet little of this finds its way into current AS/A-level specifications in mathematics, whose geometrical content has been drastically reduced over time. Similarly, the kind of geometry studied by mathematics undergraduates bears little resemblance to that studied either pre- or post-16. We refer to this issue as one of **progression.**

While the working group is optimistic about the possibility for significant improvement in teaching geometry 11-16, (which is not to underestimate the challenges to be addressed), it is far less sanguine about the state of geometry in 16-19 education. The geometrical content of the current AS/A-level specifications in pure mathematics is very small and offers little by way of progression from what has come before. But there is little point in advising content changes at this level when the whole basis of 16-19 qualifications in mathematics and all other subjects has just undergone a series of changes, the consequences of which have yet to be fully felt. Our view is that the general position of mathematics in 16-19 education needs a fundamental review before geometry can be accorded an acceptable place.

It is widely recognised that secondary schools have problems recruiting and retaining mathematics teachers. Many of those currently teaching mathematics in secondary schools are not mathematics graduates. Due to the problem of progression, it cannot be assumed that even trained mathematics graduates are adequately equipped to teach geometry in the way the working group envisages. To remedy this will require a substantial programme of well planned continuing professional development for teachers which improves both their subject knowledge in geometry and their approaches to teaching it. The current Key Stage 3 mathematics strategy provides substantial opportunities for the professional development of mathematics teachers in secondary schools and has the potential to make a valuable contribution to improvements in the teaching of geometry. But this alone will not be sufficient to improve teaching throughout the 11-19 sector.

Computer software, particularly that known as Dynamic Geometry Software, has the potential to make significant improvements in how geometry is learnt and taught. But such software is not widely available in school mathematics classrooms, as is the case with computing resources in general. In order for such resources to have maximum effect on improving the teaching and learning of geometry we need to find ways which allow talented teachers the time to develop a range of effective Information and Communication Technology based approaches. In addition to ICT, there is a need for a range of good materials to support the teaching of geometry in school. These, too, need to be carefully prepared and tried out, and that will also require time and effort.

The working group has been challenged to articulate its vision for geometry teaching. I believe that what we seek is a coherent, stimulating, rewarding and challenging geometry curriculum which is taught in a way which captures students' interest and imagination and which attracts them towards mathematics as a subject for further study. The achievement of our vision requires a significant improvement in the quality of teaching, and this has major consequences - both for the continuing professional development of teachers and for the provision of high quality supporting resources.

Summary

This report presents the findings of a broadly based working group established by the Royal Society and the Joint Mathematical Council to consider the teaching and learning of geometry in schools and colleges. The study was initiated following the publication of results of international educational comparisons, the 1999 revision of the National Curriculum for English schools 11-16, and at a time of several major policy initiatives in education.

The working group considered the rationale for a geometry curriculum, its possible content and issues concerned with its effective teaching. This report reflects its agreed views on the state of geometry teaching 11-19 and the major issues needing to be addressed to bring about improvements. It is supported by additional materials, some of which are printed here as appendices, and others of which are accessible from the Royal Society's website at www.royalsoc.ac.uk These additional materials are intended to help illustrate some of the points in the report, and to offer examples of approaches which might be taken by schools and colleges. They are sometimes attributed to an individual or groups of members and are then not claimed to represent the views of the whole group.

In order to help identify major issues raised, the report is structured around a number of agreed Key Principles. In the main body of the report these are presented together with explanations, supporting arguments and, where available, evidence. Additional information and exemplification is provided in the appendices. One or more recommendations are associated with each Key Principle.

Overall, for mathematics 11-16, we conclude that the geometrical content of the new National Curriculum, with a few adjustments, forms an appropriate basis for a good geometry education. In order for this to be achieved, however, considerable changes are needed in the way geometry is taught. It is vital that those working to improve mathematics education ensure that their work contributes significantly to improvements in geometry (as well as mathematics) teaching. Bringing about improvements in geometry teaching will require a significant commitment to a substantial programme of continuing professional development alongside the development of appropriate supporting materials.

For mathematics post-16 we conclude that there are insufficient opportunities for students to build on their 11-16 studies in geometry. Those concerned with curriculum design need to review the structure of post-16 qualifications in mathematics to ensure they provide improved opportunities for students to continue to study geometry. The provision of challenging and interesting geometry should help make mathematics a more attractive subject of study for more students. This in turn would contribute to overcoming the current shortage of those with good mathematical skills.

Key Principles

Key Principle 1: Geometry should form a significant component of the mathematics curriculum for all students from 11 to 19.

Key Principle 2: Any choice of curriculum should be underpinned by a rationale.

Key Principle 3: The geometry curriculum should maintain breadth, depth and balance, and be consistent with Key Principle 2 and the objectives in Recommendation 3.

Key Principle 4: Geometry should be given a higher status, together with a fair share of the teaching time available for mathematics.

Key Principle 5: Students in 16-19 education should have the opportunity to continue further their studies in geometry.

Key Principle 6: The assessment framework for the curriculum should be designed to ensure that the full range of students' geometrical knowledge, skills and understanding are given credit.

Key Principle 7: The most significant contribution to improvements in geometry teaching will be made by the development of good models of pedagogy, supported by carefully designed activities and resources, which are disseminated effectively and coherently to and by teachers.

Key Principle 8: It is a matter of national importance that as many of our students as possible fully develop their mathematical potential. Geometry, with its distinctive appeal, should make mathematics attractive to a wider range of students.

Recommendations

Recommendation 1: We recommend that curriculum and assessment specifications be reviewed to ensure that geometry forms a significant component of the mathematics curriculum for all students from 11 to 19.

Recommendation 2: We recommend that the title of the attainment target Ma3 of the National Curriculum be changed from 'Shape, space and measures' to 'Geometry'.

Recommendation 3: We recommend that the geometry curriculum be chosen and taught in such a way as to achieve the following objectives:
a) to develop spatial awareness, geometrical intuition and the ability to visualise;
b) to provide a breadth of geometrical experiences in 2- and 3-dimensions;
c) to develop knowledge and understanding of and the ability to use geometrical properties and theorems;
d) to encourage the development and use of conjecture, deductive reasoning and proof;
e) to develop skills of applying geometry through problem solving and modelling in real world contexts;
f) to develop useful Information & Communication Technology (ICT) skills in specifically geometrical contexts;
g) to engender a positive attitude to mathematics; and
h) to develop an awareness of the historical and cultural heritage of geometry in society, and of the contemporary applications of geometry.

Recommendation 4: We recommend that the current geometrical content of the English secondary school mathematics National Curriculum be regarded as a reasonable basis for an appropriate and rewarding geometry education for all pupils.

Recommendation 5: We recommend that the mathematics curriculum be developed to encourage students to work investigatively, demonstrate creativity and make discoveries in geometrical contexts so that students develop their powers of spatial thinking, visualisation and geometrical reasoning.

Recommendation 6: We recommend that the mathematics curriculum be developed in ways which recognise the important position of theorems and proofs within mathematics and use the study of geometry to encourage the development of logical argument appropriate to the age and attainment of the student.

Recommendation 7: We recommend that the mathematics curriculum be developed to provide ample opportunities for students to use geometry for practical problem solving through mathematical modelling in both 2- and 3-dimensions.

Recommendation 8: We recommend that the geometry curriculum be developed to give greater emphasis to work in 3-dimensions and to make better use of Information and Communication Technology (ICT).

Recommendation 9: We recommend that the use of the word 'numeracy' in government publications and announcements be replaced by 'mathematics' to ensure that geometry is accorded its rightful position.

Recommendation 10: We recommend that geometry should occupy 25% - 30% of the teaching time, and hence a similar proportion of the assessment weighting, in the 11-16 mathematics National Curriculum.

Recommendation 11: We recommend that the total time allocated to mathematics 11-16 be monitored to ensure students spend at least 3 hours a week on mathematics, so that sufficient time is given to the teaching of geometry, and to other aspects of mathematics.

Recommendation 12: We recommend that a fundamental review be made of all 16-19 mathematics provision. This should include considering how:
a) the structure and content of the current AS/A-level Mathematics and Further Mathematics specifications can better meet the needs of students and include a greater emphasis on geometry; and
b) other post-16 mathematics qualifications, such as Free Standing Mathematics Units (FSMUs) and AS-level Use of Mathematics, can enable students to have the opportunity to continue their study of geometry.

Recommendation 13: We recommend that in the 16-19 curriculum the key skill, 'Application of Number', be re-titled 'Application of Mathematics' and that the range of qualifying mathematical studies be broadened so that students continue their study of geometry.

Recommendation 14: We recommend that a review be made of the methods of assessment and examination used in mathematics at Key Stage 3, at GCSE and in post-16 qualifications to ensure that appropriate credit is given for the attainment of specific geometrical objectives.

Recommendation 15: We recommend that the relevant government agencies work together, with bodies such as the mathematics professional associations represented on JMC, to provide a coherent framework for supporting the development of teaching and learning in geometry. This will involve:
a) the recognition and development of good practice in geometry teaching through pilot studies and research;
b) the design of programmes of continuing professional development and initial teacher education;
c) the production of supporting materials; and
d) the establishment of mechanisms to provide supporting resources, including ICT.

Recommendation 16: We recommend, in terms of mathematics in general, that:
a) better publicity and information be provided to schools, students and parents about the career opportunities afforded by studying mathematics; and
b) ways be sought to encourage schools and colleges to attract more students to study mathematics post-16, particularly at A-level.

1 Introduction

This report presents the findings of a working group established by the Royal Society and the Joint Mathematical Council (JMC) to consider the teaching and learning of geometry in schools and colleges. The working group, chaired by Professor Adrian Oldknow, met fourteen times between February 2000 and May 2001. The membership of the working group is given at the front of this report and its terms of reference can be found in Appendix 1.

The study was initiated following publication of the results of international educational comparisons, the 1999 revision of the National Curriculum for English schools 11-16 and at a time of several major policy initiatives in education. Some of this background is set out in Appendices 2 and 3.

Membership of the group was carefully chosen to include those with experience in: (a) teaching mathematics in state and independent schools and colleges, in initial teacher education and in higher education; (b) conducting research in mathematics (including geometry) and in mathematics education; (c) applying mathematics (including geometry) in disciplines such as science, engineering, IT and finance, and; (d) planning and implementing mathematics curricula in Local Education Authorities (LEAs) and government agencies. A variety of groups have expectations of the mathematics curriculum and its geometrical content; some of these are considered in Appendix 4.

2 Geometry and its teaching and learning

Geometry is one of the longest established branches of mathematics. It has an extensive range of applications and we give some selective historical and cultural background in Appendix 5. Geometry has been accorded a central place in mathematical education in Western culture for a considerable period of time. One of the major achievements of classical geometry was the systematic collection by Euclid of the geometrical knowledge of the ancient Greeks. This has, until comparatively recently, formed the basis for much of the geometry taught in schools.

During a period of educational reforms in mathematics in the 1950s and 1960s some new syllabuses (sometimes called 'the new maths') were developed where the emphasis was on formal structures which were predominantly algebraic. At the same time, the range of approaches to geometry was broadened from its traditional Euclidean base (which was reduced in depth) to include the use of transformations, vectors, matrices and some topology.

In recent years many countries have been reviewing the aims, content and approach of their geometry curricula. The 1995 study by the International Commission on Mathematics Instruction (ICMI) [Mammana and Villani, 1998] revealed that no clear consensus was emerging about the outcome of these reviews. The small scale research study into the geometry curricula of a number of countries commissioned in 2000 by the Qualifications and Curriculum Authority (QCA) for England confirmed this.

Against this background the working group considered the rationale for a geometry curriculum, its possible content and issues concerned with its effective teaching. Our report sets out a number of recommendations on issues where the working group reached a consensus view. There are some matters on which the working group did not reach a conclusion, and which others may wish to pursue further. There are also some matters which the working group did not address. In order to help identify major issues raised, the report is structured around a number of agreed Key Principles. These are presented together with explanations, supporting arguments and, where available, evidence. Additional information and exemplification are provided in appendices and on the Royal Society website at www.royalsoc.ac.uk One or more recommendations are associated with each Key Principle.

3 The place of geometry in the curriculum

Key Principle 1: Geometry should form a significant component of the mathematics curriculum for all students from 11 to 19.

This is a simple proposition to express yet it has many facets. First we consider some issues about the role of geometry in education. Then we consider the relation of geometry to other aspects of the mathematics curriculum. We review some of the problems associated with teaching aspects of geometry and pave the way for other key principles which stem from this.

A valid case for the study of geometry may be made on several grounds. Geometry is a central part of mathematics, and geometrical thinking is a fundamental way to engage with mathematics. Geometry can be used to develop students' spatial awareness, intuition and visualisation. It can also be used to solve practical problems. There are many applications of geometry relevant to employment and everyday life. Other subjects in the curriculum, such as science and technology, make use of geometrical ideas and techniques. Geometry is well established in our culture and has an interesting history of its own. It has an important place in the development of aesthetics and design. It can be used to encourage the development and use of conjecture, deductive reasoning and proof. Geometry can also be used to lay foundations for further studies in mathematics.

It is our view that all of these grounds, which have often been cited in the past, remain valid reasons for the inclusion of geometry as a significant part of the current curriculum. There are additional grounds that reflect recent changes in our society.

The rapid development in a range of technologies means that citizens now and in the future will interact with a variety of forms of displayed images. These may be required by their work, be needed in order to exchange information or just be associated with leisure. A case can thus be made that geometry has a role to play within the development of citizenship in enabling students to interpret, manipulate, control and create such images.

In recent years there has been a major shift in the UK economy from manufacture to services. Associated with this has been a marked increase in demand for those with good skills in flexible thinking and the use of Information and Communication Technology (ICT), together with the ability to apply mathematics (inadequately referred to as 'numeracy' skills). A direct consequence has been the much publicised problem in recruiting and retaining mathematics teachers. In order to fulfil the skills needs of industry, commerce and the professions - including teaching - we need to encourage more students to engage positively with mathematics and to choose to continue their studies in it, or related disciplines. We believe that geometry is a subject of mathematical study which has its own appeal and satisfaction and which, well taught, could encourage more students to continue with the study of mathematics beyond 16.

Breadth of study in geometry needs to be provided to meet the demands outlined above. To ensure students also receive appropriate intellectual challenges and stimuli it is important to provide depth in a number of topics. The challenge, of course, is to do both within a fair share of the time which should be allocated to mathematics teaching.

We conclude this first Key Principle with two recommendations. We believe that geometry has declined in status within the English mathematics curriculum and that this needs to be redressed. It should not be the "*subject which dare not speak its name*".

Recommendation 1:

We recommend that curriculum and assessment specifications be reviewed to ensure that geometry forms a significant component of the mathematics curriculum for all students from 11 to 19.

Recommendation 2:

We recommend that the title of the attainment target Ma3 of the National Curriculum be changed from 'Shape, space and measures' to 'Geometry'.

4 The 11-16 curriculum

Key Principle 2: Any choice of curriculum should be underpinned by a rationale.

Here we work towards defining a set of objectives against which to evaluate the geometrical content of a curriculum. First we summarise the current position regarding the 11-16 curriculum for the maintained sector in England.

The English educational system, centrally administered by the Department for Education and Skills[1] (DfES), is organised around a number of relatively autonomous agencies and units. These include the Qualifications and Curriculum Authority (QCA), the Office for Standards in Education (Ofsted), the Teacher Training Agency (TTA) and the British Educational and Communications Technology Agency (BECTa).

Schools and colleges have already had to adapt to considerable changes in very short time scales. So, rather than attempting to develop a geometry curriculum from first principles, we have chosen to review the current curriculum.

First we consider issues concerned with the teaching of mathematics in England in secondary schools to pupils aged 11-16. The QCA published a revised version of the National Curriculum for England in 1999 for implementation in schools and colleges from September 2000. The mathematics curriculum at Key Stages 3 (ages 11-14) and 4 (ages 14-16) differs from the earlier version in a number of respects. In particular the new version details the curriculum separately for each Key Stage, whereas the earlier version combined Key Stages 3 and 4. It also divides the Key Stage 4 curriculum into two programmes of study called 'mathematics foundation' and 'mathematics higher'. The geometrical content of the new curriculum is described within the section Ma3 *Shape, space and measures*. It is described in much greater detail than in the previous version. It has been suggested that the earlier version gave scope for teachers to address the items of geometrical content found in the 1999 version. However it is our experience that some significant aspects of geometry in the new version, particularly in the higher programme at Key Stage 4, are not currently taught extensively in secondary schools.

The working group considered the way in which the 11-16 mathematics curriculum is presented, and currently examined. At Key Stage 3 there is a single curriculum for all pupils. At Key Stage 4 it is divided into two. It is anticipated that roughly half of pupils will not study many of the additional aspects of mathematics contained only in the higher programme of study. Currently the examinations for mathematics in the General Certificate of Secondary Education (GCSE) are set in three tiers: foundation, intermediate and higher. The introduction of separate programmes of study alongside the use of three examination tiers raises a number of issues. The working group chose not to consider alternative ways of packaging the curriculum as these structures have implications for the whole mathematics curriculum, not just geometry.

The original National Curriculum has gone through two sets of revisions; neither of these has provided a rationale for the content of the mathematics curriculum. In our discussions, we identified a clear need to provide a set of objectives against which curriculum content should be evaluated and which we now provide in the form of a recommendation to improve the focus, coherence and relevance of geometry teaching.

Recommendation 3:

We recommend that the geometry curriculum be chosen and taught in such a way as to achieve the following objectives:

a) to develop spatial awareness, geometrical intuition and the ability to visualise;
b) to provide a breadth of geometrical experiences in 2- and 3-dimensions;
c) to develop knowledge and understanding of, and the ability to use, geometrical properties and theorems;
d) to encourage the development and use of conjecture, deductive reasoning and proof;
e) to develop skills of applying geometry through problem solving and modelling in real world contexts;
f) to develop useful Information & Communication Technology (ICT) skills in specifically geometrical contexts;
g) to engender a positive attitude to mathematics; and
h) to develop an awareness of the historical and cultural heritage of geometry in society, and of the contemporary applications of geometry.

From the analysis of the current geometry curriculum a number of questions emerged to which the working group has worked to find answers:

- How should the geometrical content be determined?
- Does the content of the revised Ma3 curriculum form

[1] The DfES was created in June 2001. Before this, the English education system was administered by the Department for Education and Employment (DfEE)

an appropriate basis for the teaching of geometry in secondary schools at Key Stages 3 and 4?
- If not, how should it be modified?
- What suppositions are made about the time available for teaching mathematics, and its geometry component, and are these acceptable?
- What issues need to be addressed if it is to be taught effectively?
- How do assessment procedures impact on teaching?
- What are the implications for teaching geometry pre-11 and post-16?

There is no requirement for the development of the National Curriculum to be based on evaluated field trials and experiments to test feasibility. Nor are such developments necessarily linked with any associated professional development for teachers, or with any development of appropriate teaching materials or assessment. Thus, in the absence of evidence, it has to be a matter of judgement whether the geometry selected for inclusion in the content of the National Curriculum defines an attainable curriculum.

The Ma3: *Shape, space and measures* component of the 1999 National Curriculum certainly exhibits a breadth of study in geometry (see Appendix 6). It is the view of the working group, led by the experienced school teachers amongst us, that given the right circumstances it can provide an appropriate, interesting and attainable curriculum. We shall discuss what we mean by the right circumstances later in this report.

Before considering the curriculum content in greater detail we consider some recent changes in the way the mathematics curriculum is implemented and developed. The first of these is the model followed by the National Numeracy Strategy (NNS) in primary schools, which is now being extended to mathematics at Key Stage 3 in secondary and middle schools. The second is the 3 year project concerned with the teaching of algebra and geometry now being conducted by the QCA.

The National Numeracy Strategy is managed by the DfES's Standards and Effectiveness Unit (SEU). It has developed a year by year framework for teaching Key Stages 1 and 2 of the mathematics National Curriculum. This is based on work carried out in over 200 pilot schools. Associated with the detailed teaching schemes has been a large scale professional development exercise involving LEAs, headteachers, mathematics coordinators, classroom teachers, governors etc. It has thus served as a national medium for the interpretation and implementation of the established curriculum. The government has extended the work of the Strategy first into Year 7 (the year of entry to most secondary schools), and more recently into the whole of Key Stage 3. The Key Stage 3 mathematics strategy comes into national effect in September 2001 after a short pilot stage. Apart from the way it is being introduced, there are other differences between the primary and secondary strategies, among the most pressing of which is the current shortage of qualified mathematics teachers in secondary schools. We welcome the opportunities for the improvement in mathematics teaching in secondary schools which this large scale development has the potential to stimulate. In the latter stages of our work an observer from the Key Stage 3 mathematics strategy joined the working group in order to ensure better linkage between our conclusions and recommendations and the way the Key Stage 3 strategy will implement the teaching of Ma3. Brief examples from the current framework for mathematics in Years 7, 8 and 9 appear in Appendix 7, and there are links to the full document on the Royal Society website at www.royalsoc.ac.uk

An earlier working group of the Royal Society and JMC produced a report on the teaching of algebra pre-19 [Royal Society / JMC 1996]. The DfES is now supporting a 3 year study by the QCA into the teaching of Algebra and Geometry. This has already commissioned international studies into the teaching of those aspects of mathematics. See also Appendix 2 for a brief discussion of international trends. We welcome the opportunity which this new project offers to implement our recommendations for the teaching of geometry. We also welcome the extended time scale for this project.

Recommendation 4:

We recommend that the current geometrical content of the English secondary school mathematics National Curriculum be regarded as a reasonable basis for an appropriate and rewarding geometry education for all pupils.

5 The development of the curriculum

Key Principle 3: The geometry curriculum should maintain breadth, depth and balance, and be consistent with Key Principle 2 and the objectives in Recommendation 3 and Appendix 11.

In reviewing the curriculum we paid particular attention to a number of important aspects of geometry related to our recommended set of objectives.

Through tackling, and solving, problems in geometry (both closed and open ended) pupils can develop 'thinking skills' of reasoning, enquiry (which includes problem posing and conjecturing) and creativity. They can also develop their geometrical intuition and extend their powers of visualisation and spatial thinking. These aspects are considered further in Appendix 8.

An important aspect of geometry is concerned with the development of deductive reasoning and proof. Of course proof is not confined to geometry alone, and there can be interactions, such as algebraic results proved geometrically and vice versa. However the use of geometry as a vehicle for the development of the understanding and use of deductive reasoning has received relatively little emphasis in the English school curriculum over the last 30 years.

For a variety of reasons, the whole issue of proof within school geometry has become emotive. In some minds it is associated with a particular style of teaching and examining sometimes pejoratively, and erroneously, associated with the name Euclid. In others, it is regarded as the essential difference between mathematics and the experimental sciences, and as an essential tool for the further study of mathematics. The working group has had many interesting discussions about the place of geometrical proofs within mathematics, particularly at Key Stages 3 and 4. We have also received advice from individuals and bodies representing many shades of opinion - with a strong representation in favour of geometrical proof from correspondents in Higher Education and from some school teachers. We have concluded that it is important for all students to encounter proof during their study of geometry, while also recognising that some aspects of proof may be more accessible in other mathematical contexts. For a discussion of what we mean by proof see Appendix 9.

There is no suggestion here to attempt an axiomatic approach to school geometry. Indeed we note that such attempts have been made, unsuccessfully, in the past. Rather we are arguing for the use of logical argument, which builds upon what is already known by the pupil in order to demonstrate the truth of some geometrical result, possibly one conjectured by the pupil after conducting a well chosen experiment. The results concerned (i.e. the theorems) should be chosen as far as possible to be useful, interesting and/or surprising. The level of sophistication expected in the logical argument will depend upon the age and ability of the pupil concerned, and the proof produced might equally be called an 'explanation' or 'justification' or 'reason' for the result. Many pupils may never reach the level of providing formal proofs of results (although the more able should), but all should understand deductive reasoning and that it is more than simply stating a belief or checking that the result is valid in many specific cases. Encouraging pupils to be critical of their own, and their peers', explanations will help them develop the sophistication and rigour of their arguments. The emphasis at all times should be on understanding, and analysing a proof of a standard theorem has a positive role in understanding too. Without doubt, the end result of a proof at this level should be an understanding of why the result is true, not simply that a formal argument proves it. However, we accept that it is not an easy matter to determine how to achieve this with each pupil and each result and that a careful choice of approach will be needed. Some examples of proof in a variety of areas and styles appear in Appendices 9 and 11.

We are aware that there are considerable difficulties to be overcome in achieving our objective "*to encourage the development and use of conjecture, deductive reasoning and proof*" (Recommendation 3d). We do not have a successful experience base to fall back on, nor have we found that other countries have positive lessons to offer. We have found that many teachers currently in post or in training do not have experience in using geometrical reasoning themselves. We consider the implications of these issues in Key Principle 7 below.

Mathematical reasoning is one strand of a fundamental, and unusual, area of the mathematics National Curriculum called *Using and Applying Mathematics*. We now consider its other strands which relate to communication and problem solving. In previous versions of the National Curriculum this area was described as a separate component, Ma1. In the current version it has been integrated within each of the other three components, including Ma3 *Shape, space and measures*. The working group accepts that it is important for all students to appreciate the power of mathematics in the way it is applied in modelling important phenomena and solving practical problems - and that this applies equally in geometry as in other areas. We have concluded that it is important for all students to experience the applicability of geometry through engaging in mathematical modelling in 2- and 3-dimensions. Geometrical ideas are used in many models which pupils will encounter and use in the future. We are aware of the possibly confusing nature of the word 'model' in this context. By a 'mathematical model' we mean a representation through the language of mathematics of a real world problem. 'Modelling' is the process of translation into mathematics, usually involving simplification and idealisation. This modern use of the word recognises that this process of translation is itself a skill to be learned. It also recognises that the resulting mathematics will never be a perfect description of the original problem. We give some examples of the ways in which geometry can be used in school to model familiar situations in Appendix 10.

Another important feature of the geometry curriculum is that it provides opportunities for pupils to draw sketches, diagrams and accurate representations. We give just a few examples. Pupils can learn to use the properties of figures, such as isosceles triangles, rhombuses and kites, to develop mathematically exact constructions on paper with straight-edge and compass. They can explore whether sets of the same figures can be arranged to tile the plane. They can explore and apply properties of standard figures, as well as constructions, on computers using suitable geometry software. They can produce plane sections of 3-D objects to apply their knowledge of 2-D figures in solving problems in 3-D. They can sketch perspective drawings of 3-D objects from different viewpoints. They can make nets from which to construct 3-D solids.

The section on Transforming Secondary Education in the Green Paper, '*Schools, Building on Success*' of February 2001 includes the following:

> *4.29 The goals of our Key Stage 3 strategy are to ensure that by age 14, the vast majority of pupils have: .... Learnt how to reason, to think logically and creatively and to take increasing responsibility for their own learning.*

Responsibility for the development of pupils' thinking skills now comes within the 'Teaching and Learning in the Foundation Subjects' component of the government's Key Stage 3 strategy. The working group welcomes this recognition of the importance of thinking skills and recommends that the national Key Stage 3 strategy makes use of the geometry component of the mathematics curriculum for the development of such skills.

So, consistent with our stated objectives, the working group advocates striking a balance between the creative, deductive and applicable aspects of geometry.

Recommendation 5:

We recommend that the mathematics curriculum be developed to encourage students to work investigatively, demonstrate creativity and make discoveries in geometrical contexts so that students develop their powers of spatial thinking, visualisation and geometrical reasoning.

Recommendation 6:

We recommend that the mathematics curriculum be developed in ways which recognise the important position of theorems and proofs within mathematics and use the study of geometry to encourage the development of logical argument appropriate to the age and attainment of the student.

Recommendation 7:

We recommend that the mathematics curriculum be developed to provide ample opportunities for students to use geometry for practical problem solving through mathematical modelling in both 2- and 3-dimensions.

We now consider what might be missing from the current curriculum. The first matter we identified is the need for much greater attention to 3-D geometry at each stage of the curriculum for all pupils whatever their ability. It is simplistic just to note that we live in a 3-D world and need to be able to develop the geometrical skills to represent 3-D objects and to solve problems involving them. Clearly 3-D modelling is of great importance in a wide range of disciplines, such as science, engineering and design. We now come into contact with a much wider range of 2-D representations of 3-D objects than was previously the case. Spatial awareness, powers of visualisation and realistic means of applying geometry cannot be developed successfully without paying greater attention to work in 3-D. So we propose that in the 11-16 curriculum students should extend their understanding, skills and knowledge of geometry in the plane to solve problems in 3-D. Of course, some 3-D work relies on 2-D results which will need to be established first.

The revision of the National Curriculum by QCA in 1999 gave the opportunity for greater exemplification of the ways in which Information and Communication Technology impacts on many subjects and their teaching. Yet there is very little specific reference to the use of ICT in the mathematics National Curriculum in general, and in geometry in particular. Geometrical software is now widely used in, for example, engineering and design. Through government and commercial initiatives many secondary schools and colleges have acquired powerful Computer Aided Design and Computer Aided Manufacture (CADCAM) packages for use in teaching Design and Technology. By contrast relatively few schools have access to software for teaching geometry in mathematics. Yet by using such software in appropriate ways, pupils can apply their ICT skills to increase their knowledge and understanding of geometry. The software also provides them with the opportunity to acquire and practise geometrical skills. Opportunities occur when pupils create, analyse and interpret dynamic spatial images; make and test conjectures about geometrical relationships that they can manipulate; and record and present the results of their investigations.

As with any approach to teaching, the educational use of ICT needs to be well thought through and carefully planned. The TTA has produced documentation to accompany the current programme of lottery funded ICT training for all teachers in which it emphasises the importance of a critical approach to the use of ICT. This expects teachers to know where, when and how to apply ICT to enhance the teaching and learning of their subjects. This advice is particularly important in geometry where a variety of approaches is needed including mental, practical, and ICT enhanced work. Increasingly powerful software is becoming available in education, such as that designed for simulations in science and geography, much of which relies on sophisticated mathematical algorithms. Pupils and teachers in all subjects need to be cautious about accepting computer produced results without question, and mathematics is probably the subject best placed in the curriculum in which to engender a critical approach. In teaching geometry, caution is particularly needed to avoid making assertions based solely on computational illustrations.

Thus the working group would like to see further development of the curriculum with respect to work in 3-D and the use of ICT. Appendix 11 on 3-D geometry gives examples of five topics that are suitable for schools. We recognise that this will have implications for resources, materials, assessment and teachers' professional development, as will the **effective** teaching of proof, modelling, problem solving and other aspects of geometry. In many respects we need to develop a completely new pedagogy in geometry. We consider such issues further below. We also recognise that we are advocating an extension of the current curriculum, even before it has been fully implemented. Conscious of the potential criticism for proposing to extend an already crowded curriculum we address the issue of time allocation for mathematics below. Experienced teachers have developed their own mechanisms for setting out the curriculum in such a way that links can be made and time used most effectively. In Appendix 12 we include an extract from a possible framework for the extended geometry curriculum devised by some members of our working group.

Recommendation 8:

We recommend that the geometry curriculum be developed to give greater emphasis to work in 3-dimensions and to make better use of Information and Communication Technology (ICT).

6 Status and allocation of time to geometry

Key Principle 4: Geometry should be given a higher status, together with a fair share of the teaching time available for mathematics.

Recently there has been a tendency to replace the word 'mathematics' with 'numeracy', as if the two were equivalent. This has sent out mixed messages about the relative importance of different aspects of the mathematics curriculum. For example, within the pages of the DfES *The Standards Site* on the Internet (www.standards.dfee.gov.uk/numeracy/) the following description of the National Numeracy Strategy may be found:

> *Framework for teaching mathematics*
> *The Numeracy Framework helps teachers raise numeracy standards nationwide by providing them with a set of yearly teaching programmes, key objectives and a planning grid.*

Similarly the introduction to the government Green Paper, *Schools: Building on Success*, published in February 2001, contains the following:

> *Every secondary age pupil must be competent in the basics of literacy, numeracy and ICT and experience a broad curriculum beyond.*

We are concerned that this concentration on numeracy should not result in the sidelining of geometry.

Recommendation 9:

We recommend that the use of the word 'numeracy' in government publications and announcements be replaced by 'mathematics' to ensure that geometry is accorded its rightful position.

While accepting that the area called Ma2 *Number and algebra* in the secondary school curriculum should have the greatest amount of teaching time, we regard 25% of the available mathematics time as the minimum necessary for the teaching of geometry in Ma3. We are concerned about reports from some secondary schools that there has been an erosion in the total time available for teaching mathematics, particularly in Key Stage 3 - perhaps exacerbated by teacher shortages. Primary schools following the NNS have a daily mathematics lesson which, by the end of Key Stage 2, lasts one hour. Secondary schools following the new Key Stage 3 strategy have been given guidelines for the time to be allocated to mathematics - at least 3 hours per week. Members of the working group have also expressed the view that if mathematics is to have parity of esteem with the other core subjects of science and English then it should be available as a double award at GCSE.

We do not have specific proposals to make about the teaching of geometry in primary schools. The National Curriculum at both Key Stage1 (pupils aged 5-7) and Key Stage 2 (pupils aged 7-11) has the Ma3 *Shape, space and measures* component. The NNS's *Framework for teaching mathematics from Reception to Year 6* provides an interpretation of this within the framework of the daily mathematics lesson. Provided that this curriculum is effectively implemented, then pupils transferring from primary schools to Year 7 in secondary schools should have a suitable basis on which to develop their study of geometry.

The working group is aware that the effective teaching of the secondary school geometry curriculum which it advocates is likely to require rather more time for geometry than is currently normally the case. The renaming of Ma3 to 'Geometry' should imply that the work on non-geometrical measures, such as time and speed, is relocated in Ma2. Some aspects of Ma2 *Number and Algebra* could be developed within geometrical contexts, such as Pythagoras's Theorem.

Questions have been raised about the time, and assessment, allocation to Ma4 *Handling Data*, and even as to whether it should be part of the mathematics curriculum at all. However we do not wish to make any recommendations in respect of the content of this, or other, parts of the mathematics curriculum. That is not to duck the issue but to record that it is for others to assess the strength of our claims for geometry against those of other parts of the curriculum. We have already pointed to the lack of a sound experience base for an appropriate pedagogy for significant aspects of the geometry curriculum, for which there is an urgent need. It could well be that with the right approach, supported by appropriate materials and resources including ICT, the teaching of geometry could also be made more efficient. In summary we believe that a broad, coherent and demanding geometry curriculum can be effectively taught within a fair and reasonable time allocation. This may require some review of the balance between the components of the mathematics curriculum. It will certainly require the development of more effective and efficient teaching approaches.

Recommendation 10:

We recommend that geometry should occupy 25% - 30% of the teaching time, and hence a similar proportion of the assessment weighting, in the 11-16 mathematics National Curriculum.

Recommendation 11:

We recommend that the total time allocated to mathematics 11-16 be monitored to ensure students spend at least 3 hours a week on mathematics, so that sufficient time is given to the teaching of geometry, and to other aspects of mathematics.

7 Geometry 16-19

Key Principle 5: Students in 16-19 education should have the opportunity to continue further their studies in geometry.

The government has implemented reforms in the post-16 sector called 'Curriculum 2000'. Students are now encouraged to follow a programme of study which includes key skills, among which is the 'Application of Number'. For most students this will be the only course of mathematics they study post-16. Consistent with our first Key Principle, we propose that its title should be changed and its content extended so that students study material from a wider range of topics in mathematics, including geometry. There should be more compulsory elements of geometry which are assessed through tests, and which make explicit the opportunities to develop geometrical ideas in greater depth for inclusion in the portfolio.

The QCA have also recently revised their criteria for the mathematics General Certificate of Education Advanced Subsidiary and Advanced Level (AS- and A-level). Awarding bodies have now produced specifications that are being taught for the first time in the current academic year. The geometry in the compulsory part (core) of pure mathematics for A-level consists of a very small amount of coordinate geometry (lines and circles), some trigonometry and some elementary work with vectors. Within the current framework there is little scope for more geometry in the core, but more use could be made of geometrical contexts, say in the application of calculus. The working group doubts that the current geometrical content of A-level mathematics forms a suitable foundation for those students who go on to study science or engineering. In particular there should be greater emphasis on work in 3-D.

The working group has discussed the possibility of introducing one or more optional modules at AS- and A-level, outside the core of pure mathematics, which could include extensions in geometry. Potential drawbacks of such a solution would be the increase in the variety of routes to an award - leading to problems of comparability of standards, and also a greater variety of mathematical backgrounds of students taking the same course in higher education. We do not have an instant solution to propose with regard to improving the geometrical content of AS- and A-level mathematics in their current format. When a fundamental review of these qualifications takes place the working group recommends that careful consideration be given to extending the amount of geometry in the A-level core. Candidates for an extension to such content include plane curves, such as conics, further vector geometry and a greater emphasis on parametric representations.

We would expect a greater emphasis to be given to the important role of coordinate geometry as a link between algebra, graphs and functions, and calculus.

More generally there is reason to believe that the existing choice of optional modules (mainly in mechanics or statistics) does not meet the needs or interests of all potential candidates for A-level mathematics - such as those with an interest in aesthetics, or an intention to pursue careers in the IT industry. Currently about 60 000 of the c.230 000 A-level candidates enter for A-level mathematics. The uptake in mathematics at this level might be increased if there was a wider choice of modules, one or more of which included interesting geometry. We understand that evidence is beginning to emerge that the number of students taking AS-level mathematics as a fourth subject in the lower sixth-form (Year 12) and then not choosing to progress to A-level mathematics in the following year is higher than anticipated. However the working group is aware that much of this is speculation, and so we recommend that research be undertaken into the mathematical needs and interests of post-16 students, and the implication for the curriculum. Our conclusion is that both the structure and the content of AS- and A-level mathematics are in need of a fundamental review.

Candidates who wish to extend their post-16 study of mathematics can study A- or AS-level Further Mathematics, although numbers doing so have been dwindling. In the 1980s around 12 000 students were taking 'double mathematics' each year at A-level, whereas in the late 1990s this had levelled out at just over 5000 taking A-level Further Mathematics, and around 2000 taking AS-level Further Mathematics – despite the growth in the numbers within the A-level cohort. A major factor has been the problem of maintaining financially viable group sizes in schools and colleges. The Gatsby Foundation is supporting a project to make these courses more widely available to students through distance learning. We welcome such initiatives to encourage greater take up of these courses, as students taking such qualifications are much better placed for success in undergraduate studies in mathematics, physics and engineering. Similar arguments apply about making these courses more interesting and challenging by including both more geometry and the greater use of geometrical contexts.

There is now a range of post-16 qualifications called 'Free Standing Mathematics Units' (FSMUs) which are designed to support students in their other studies and which should enable more students to pursue mathematics post-GCSE. The FSMUs are available at three levels. There are some FSMUs which include geometry, but none at level 3, the advanced level. We recommend that a geometry unit be developed at level 3. In Autumn 2001 a new AS-level qualification called 'Use of Mathematics' is to be introduced, based on advanced level FSMU modules. The working group had understood that this new qualification was intended for students who would not otherwise take a post GCSE mathematics course, such as those specialising in the arts, humanities and social sciences. At the time of writing, the qualification is still in development but it appears that it will now be predominantly a course in mathematical modelling with no geometrical content at all. Thus there would still appear to be a gap in the market for an AS-level qualification in mathematics which will appeal to students specialising, say, in the arts and humanities and for whom geometry might be an attractive element of study.

The working group believes that there is still more to be done to ensure that there is a sufficient range of level 2 and 3 mathematics qualifications to attract greater numbers of students to continue their studies in mathematics post-16. We recommend that the range of level 3 mathematics qualifications be reviewed to ensure that students have the opportunity to study geometry further. In particular the cultural, aesthetic and historical aspects of geometry, such as the development of perspective, should be of considerable appeal to many of those students from the arts and humanities who currently drop mathematics.

The Royal Society website provides links to current specifications of post-16 mathematics qualifications.

We consider that Curriculum 2000 may have an adverse effect on mathematics, stemming partly from its complexity and rigidity. However as major changes to 16-19 education are currently being implemented, it is not the time to make any detailed recommendations with respect to specific mathematics qualifications. Thus we make the following general recommendations.

Recommendation 12:

We recommend that a fundamental review be made of all 16-19 mathematics provision. This should include considering how:

a) the structure and content of the current AS/A-level Mathematics and Further Mathematics specifications can better meet the needs of students and include a greater emphasis on geometry; and

b) other post-16 mathematics qualifications, such as Free Standing Mathematics Units and AS-level Use of Mathematics, can enable students to have the opportunity to continue their study of geometry.

Recommendation 13:

We recommend that in the 16-19 curriculum the key skill, 'Application of Number', be re-titled 'Application of Mathematics' and that the range of qualifying mathematical studies be broadened so that students continue their study of geometry.

8 The role of assessment

Key Principle 6: The assessment framework for the curriculum should be designed to ensure that the full range of students' geometrical knowledge, skills and understanding are given credit.

We do not believe that many of the geometrical objectives in Recommendation 3 can be adequately assessed within the current framework of timed tests and examinations. Indeed, the current assessment framework is one of the major reasons why important aspects of geometry, such as work in 3-D, geometrical reasoning and the use of ICT have not been given sufficient attention in classrooms. It is only natural that teachers concentrate on aspects of a curriculum which carry the greatest assessment weighting - especially within the current climate of target setting in schools.

A number of questions set in national tests and examinations which are apparently about geometry are in practice mainly exercises in algebra. We would like to see national tests and examinations incorporate questions which test geometrical reasoning and applications of geometry. If, as we suspect, there is little experience in doing so, then we recommend that the QCA commissions work to develop more appropriate forms of examination questions in geometry.

GCSE examinations in 2003 will contain a compulsory course work element. This will consist of two extended tasks each contributing 10% of the total marks. One of these has to be from Ma4 *Handling data*. The other is to demonstrate skills from *Using and applying mathematics* in the context of either Ma2 *Number and algebra* or Ma3 *Shape, space and measures*. The working group welcomes to some extent this extension of assessment techniques but queries the rationale used to make Ma4 compulsory and not Ma3. We consider that the opportunity afforded for extended work in Ma3 would be an effective way to ensure that some of our objectives for geometry teaching are more effectively fulfilled. We are concerned that teachers faced with a choice between Ma2 and Ma3 may reject geometry in favour, say, of algebra - perhaps because of their own subject confidence or because they judge the more algorithmic nature of some forms of algebraic enquiry to be a 'safer bet'. Thus we recommend that GCSE mathematics should include some compulsory course work in geometry.

The examinable course work element in the new AS/A-level mathematics course is almost entirely restricted to the applications, such as statistics and mechanics. If, as we recommend, greater opportunity is afforded to students to extend their study of geometry on these courses then a review of the appropriate means of assessment is also needed. By contrast the FSMUs have their own forms of assessment - usually 50% examination and 50% coursework. Many units specify and assess the use of appropriate ICT.

If the assessment framework for the curriculum in geometry can be developed to include both better examination questions and a reasonable contribution from extended course work, then we believe that teachers will also be encouraged to develop formative assessments in geometry for their students. It is a wider question than this working group's remit to consider whether teachers' assessments of students' progress should contribute to National Curriculum assessment and to public examinations. Such a change would require the kind of reinstatement of teachers' professional judgements which is discussed in the recent government Green Paper [DFEE, 2001].

Recommendation 14:

We recommend that a review be made of the methods of assessment and examination used in mathematics at Key Stage 3, at GCSE and in post-16 qualifications to ensure that appropriate credit is given for the attainment of specific geometrical objectives.

9 Teaching of geometry

Key Principle 7: The most significant contribution to improvements in geometry teaching will be made by the development of good models of pedagogy, supported by carefully designed activities and resources, which are disseminated effectively and coherently to and by teachers.

We did not enquire further into the impact of different forms of school organisation, such as 'setting' and mixed-ability teaching, nor did we reach a consensus about the issues arising from providing a more inclusive curriculum or from providing some more differentiated curricula. We anticipate that pilot studies in good practice may provide some helpful guidance in respect of these issues.

We now turn to the description of the 11-16 National Curriculum. The phraseology used throughout is *"pupils should be taught to…".* The National Curriculum handbook sets out in some detail what should be taught, but not why, or how. A good deal of scope for interpretation still rests with the teacher. We are aware that a tendency has recently developed in teaching the mathematics National Curriculum which breaks it down into a large number of very limited objectives - sometimes known as 'bite-sized chunks'. Such an approach can, and often does, result in fragmentation and in the failure to develop important links between curriculum areas. We believe that the successful implementation of the Ma3 component in the classroom will only be achieved if teaching programmes are focused and coherent, and if they develop links within geometry and mathematics generally where appropriate. In Appendix 13 we give some examples of ways in which aspects of the geometry curriculum could be integrated within a particular theme, and also where aspects of geometry could be linked with other areas of mathematics such as algebra and handling data.

Individual teachers implement the curriculum by planning schemes of work and lessons for their classes. So it is a matter of the greatest importance to ensure that teachers have the necessary information, skills and resources to interpret the aims and objectives of the curriculum. Recent moves to ameliorate problems of recruitment and retention of teachers, together with the government's intention to modernise working practices in health and education, mean that there is now a much more favourable climate for improving the system of teachers' continuing professional development (CPD). The working group welcomes the declared intention to provide CPD support to improve and update teachers' subject knowledge and related pedagogy.

Research, such as that reported by Hoyles and Healey, has confirmed the views of experienced teachers in schools that there are many teachers of mathematics who have large gaps in their knowledge of geometry. Similarly, we believe that there are also many teachers who have been taught geometry through styles of teaching which we would not advocate as appropriate. Thus our view is that in respect of geometry teaching there is a need for a significant CPD initiative. Government is giving greater attention to spreading good practice between teachers and schools through initiatives, such as the beacon schools. The DfES (whilst still DfEE) recently launched its CPD strategy. This has its own website at: www.dfee.gov.uk/teachers/cpd where CPD initiatives are presented under the strap line "*Learning from each other... Learning from what works*". The working group welcomes this approach. We regard it as vital that pilot studies should be carried out without delay to identify and enhance good practice in the teaching of geometry. At the same time planning should take place for a national system of provision for CPD in geometry and its teaching. This could be within the framework of the Key Stage 3 strategy. One idea which has received some support is the provision of two week geometry summer schools for serving teachers, teachers currently in training and those about to embark on a course of initial teacher education (ITE). Financial inducements may be needed to encourage attendance in vacation time. By concentrating on subject knowledge in geometry, as well as its teaching, such courses should generate enjoyment of mathematics and thus help as one part of the long term process of sustaining or renewing teachers' enthusiasm for it.

New graduates entering courses of initial teacher education have very varied backgrounds in geometry. Many will have experienced little, if any, geometry at sixth form or university level. Within the current statutory curriculum for the initial training of secondary mathematics teachers there is little scope to provide the rich overview of geometry that we believe is essential for effective teaching. Further professional development for teachers early in their career is essential, but is most likely to concentrate on the development of their teaching skills. So it is important that, in parallel with developments in CPD to support the teaching of geometry, there is a recognition of the need to improve the geometrical background of those intending to enter mathematics teaching during, or before, their initial training.

In order to support the developments in the effective teaching of geometry which we seek there is a need for a variety of materials in both printed and digital form, as well as resources such as models, posters, activity kits, videos, libraries of digital images, computer software and the like. Some of this already exists and we provide a far from exhaustive list of these in Appendix 14. An important activity will be to review the current provision and to develop new materials and resources as appropriate.

Geometry teaching outside primary schools can be, and has been, conducted with a minimal amount of equipment - such as a stick of chalk and a piece of string (echoing images of ancient Greeks drawing in the sand). Our view is that teachers should now have at their disposal an appropriate variety of equipment from which to select, depending on fitness for purpose. In particular we wish to see the potential of ICT realised in supporting the teaching and learning of geometry. There is already software available, such as for dynamic geometry (DGS), but its use is not widespread. Many schools do not have licences for the software. There is also a need for the development of additional software, such as to support work in 3-dimensions. Increasing numbers of schools and colleges are now being equipped with interactive whiteboards - where a computer image is projected onto a touch sensitive screen. This medium has considerable potential for interactive whole-class teaching of geometry. We would like to see the funding to schools for ICT being used more effectively to support the geometry curriculum.

The mathematics professional associations have a key role to play in each of these developments in partnership with the newly formed General Teaching Council (GTC) and other bodies such as the Royal Society, Higher Education institutions, the National Numeracy Strategy, QCA, Ofsted, TTA and BECTa.

Recommendation 15:

We recommend that the relevant government agencies work together with bodies, such as the mathematics professional associations represented on JMC, to provide a coherent framework for supporting the development of teaching and learning in geometry. This will involve:

a) the recognition and development of good practice in geometry teaching through pilot studies and research;
b) the design of programmes of continuing professional development and initial teacher education;
c) the production of supporting materials; and
d) the establishment of mechanisms to provide supporting resources, including ICT.

10 Improving the take up of mathematics

Key Principle 8: It is a matter of national importance that as many of our students as possible fully develop their mathematical potential. Geometry, with its distinctive appeal, should make mathematics attractive to a wider range of students.

In launching UK Maths Year 2000, the Prime Minister made clear the importance of mathematics in the education of those creative and flexible thinkers on whom our national economic prosperity will depend. The demands of commerce and industry for articulate graduates with mathematical skills far outstrips the supply of graduates in mathematics and related fields. One consequence of this is the current severe shortage of new teachers, especially for mathematics in secondary schools. For a variety of reasons, insufficient numbers of our ablest students are choosing to pursue mathematics as a specialism following both GCSE and AS/A-level. There is no universal panacea. So it is vital that we take any opportunity to review where and how the subject could be made more interesting, attractive, relevant, challenging, rewarding and engaging to all students. We are convinced that geometry has a lot to offer in this respect. For some students it may be the logical aspects which are the most appealing, for others it may be the visualising, or the modelling, or the historical and cultural, or the visual and aesthetic aspects.

In general we believe that students are not given enough information about the importance of mathematics in the world of work, and the significant advantages a mathematical education can bestow in terms of employability. The ways in which the performance of secondary schools and colleges are published through examination results takes no account of the relative national economic importance of some subjects over others. So, for example, there is a positive incentive for institutions to persuade students to choose subjects in which it is easier to achieve high grades at A-level than those subjects judged harder, which include mathematics and physics.

Overall the profile of mathematics needs to be higher in schools, colleges and universities if we are to attract more students at all levels of attainment to realise their potential in the subject. This means that still more needs to be done to improve the status of mathematics teaching, and to attract (and retain) good recruits. It also means that students should be made more aware of the relationships between mathematics and the other subjects they study. We believe that geometry is a good vehicle for achieving this aim.

Recommendation 16:

We recommend, in terms of mathematics in general, that:

a) better publicity and information be provided to schools, students and parents about the career opportunities afforded by studying mathematics; and

b) ways be sought to encourage schools and colleges to attract more students to study mathematics post-16, particularly at A-level.

11 Conclusion

For mathematics 11-16, we have concluded that the geometrical content of the new National Curriculum, with a few adjustments, forms an appropriate basis for a good geometry education. In order for this to be achieved, considerable changes are needed in the way geometry is taught. It is vital that those working to improve mathematics education ensure that their work contributes significantly to improvements in geometry (as well as mathematics) teaching. Bringing about improvements in geometry teaching will require a significant commitment to a substantial programme of continuing professional development, together with the development of appropriate supporting materials.

For mathematics post-16 we have concluded that there are insufficient opportunities for students to build on their 11-16 studies in geometry. Those concerned with curriculum design need to review the structure of post-16 qualifications in mathematics to ensure they provide better opportunities for students to continue to study geometry. More generally there is at present a severe shortage of those with good mathematical skills - and the provision of challenging and interesting geometrical content and contexts should be a valuable means to make mathematics a more attractive subject of study for more students.

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Glossary

AS level	Advanced Subsidiary Level – a qualification between GCSE and A-level
BECTa	British Educational and Communications Technology Agency
CADCAM	Computer Aided Design and Computer Aided Manufacture
CPD	Continuing Professional Development
DfEE	Department for Education and Employment (now replaced by the DfES)
DfES	Department for Education and Skills
DGS	Dynamic Geometry Software
FSMU	Free Standing Mathematics Unit
GCSE	General Certificate of Secondary Education
GTC	General Teaching Council
HE	Higher Education
ICMI	International Commission on Mathematics Instruction
ICT	Information and Communications Technology
IT	Information Technology
ITE	Initial Teacher Education
ITT	Initial Teacher Training
JMC	Joint Mathematical Council of the United Kingdom
KS	Key Stage (of the National Curriculum)
LEA	Local Education Authority
Ma3	"*Shape, space and measures*" component of the mathematics National Curriculum
NC	National Curriculum
NNS	National Numeracy Strategy
NOF	New Opportunities Fund (a Government funding initiative)
Ofsted	Office for Standards in Education
QCA	Qualifications and Curriculum Authority

QTS	Qualified Teacher Status
SEU	Standards and Effectiveness Unit (of the DfES / DfEE)
TIMSS	Third International Mathematics and Science Study
TTA	Teacher Training Agency

Appendix 1: The working group

1 Terms of reference

a) to make recommendations about teaching methods and the content of the curriculum with relation to the topic of geometry taught to pupils aged 11 to 19, in order to inform discussions of any future curriculum revisions;

b) to take into account evidence about current competence and future needs in geometry among different groups of pupils;

c) to examine the influences on pupils' experience of geometry within all aspects of the pre-university education system, including vocational provision. Whilst focusing predominantly on 11-19 provision, account should be taken of the experiences of pupils pre-11 to inform the study where appropriate.

2 Evidence received by the working group

We are grateful to the following organisations and individuals who made written submissions to the group:

Association of Teachers of Mathematics (ATM)
British Society for Research into the Learning of Mathematics (BSRLM)
Heads of Departments of Mathematical Sciences (HoDoMS)
Institute of Physics
Institution of Structural Engineers
London Mathematical Society (LMS)
Mathematical Association (MA)

Afzal Ahmed
Ron Allpress
Vernon Armitage
Keith Austin
Brian Bolt
David Burghes
Hugh Burkhardt
Bob Burn
Tandi Clausen-May
Randal Cousins
CTJ Dodson
David Fairlie
Chris du Feu
David Fielker
Ruhal Floris
Doug French
Tony Gardiner
Howard Groves
Keith Hamflett
Adrian Hill
Celia Hoyles
Graham Jameson
Gerry Leversha
John Mason (Open University)
John Mason (University of Huddersfield)
Michael McIntyre
Les Mustoe
Alice Rogers
Kenneth Ruthven
Stuart Rowlands
Peter Saunders
Mike Savage
Robin Scott
Alan Selby
Peter Shannon
John Sharp
John Silvester
Patricia Smart
Tony Sudbury
Garth Swanson
Patricia Watson
Nick Woodhouse
Derek Woodrow
Nicholas Young

3 Brief extracts from evidence received for consideration by the working group

prepared by John Rigby

Submissions were received from organisations and individuals, by invitation from the chairman and from other members of the working group. The most widely held view was that basic plane geometry, including such things as proofs of angle and circle theorems, should be reinstated as a major part of the high school mathematics curriculum. But these extracts have been selected to show the range of suggestions made and of views expressed; thus some minority opinions have been quoted in detail, whilst other valuable contributions have been omitted when they only serve to reinforce the majority views about the importance of proof and of geometrical and spatial intuition.

"The geometry that could figure in schools falls broadly into two types: geometry as a study of spatial and logical relationships and geometry as the visualisation of the real-world or the visual illustration of other parts of mathematics and of science. Both have their place, but it is work at the appropriate level in the first type that enables progress to be made in the other. ... It is essential that some logical relationships between results are demonstrated."
Heads of Departments of Mathematical Sciences (HoDoMS)

"Many more pupils could reach level 8 of the National Curriculum by the end of Key Stage 3 if expectations were higher. ... There are a number of topics which ought to be

considered by many more pupils at an earlier stage. The properties of parallel lines, a simple proof of the angle sum of a triangle and the properties of polygons all seem obvious topics for year 7. The theorem of Pythagoras should be encountered by a wide range of pupils in year 9 and by the abler ones in year 8. On the other hand there is no reason why a formal knowledge of congruent triangles should be developed before year 9, beyond a simple intuitive understanding of the word congruent."
The Mathematical Association

"[The members of the Education Training and Examinations Committee] feel that a basic ability in geometry is vital for engineers, particularly structural engineers, and that the standard of geometry being taught in schools should be raised. Geometry is a useful discipline not only in that it teaches rigour of thought but also develops the ability of a student to think spatially. ... Good spatial awareness can be engendered at all levels of education without recourse to mathematical justification at the initial stages."
The Institution of Structural Engineers

"The reform in the teaching of Euclid meant its removal because everyone assumed that Euclid could only be taught in the way it was taught by the Victorians. What we have now is 'shape and space' and the banishment of proof. ... Greek geometry ought to be accessible to every secondary school pupil, not as an exercise in rote learning ... but as an induction into one of the foundational disciplines of the mind."
Research Fellow, University Department of Mathematics and Statistics

"We live in a three dimensional world, a world full of objects which interact with each other which is a far cry from the geometry of points, lines and triangles common to most school text books. ... What we teach under the name of 'geometry' should help our pupils to a better understanding of the space around them, of the structures of the buildings, the bridges, the cranes, furniture and the thousands of machines and mechanisms that influence their daily lives."
University lecturer in mathematics education

"Three positive lessons emerge from this study [of conflict between mathematics graduates' proof behaviours and their stated beliefs about proof]:
(i) the importance of establishing key mental schemas (such as that associated with logical deduction, and with precedence-respecting logical hierarchies) as early as possible - probably beginning well before the age of 15-16;
(ii) the fact that it requires both systematic effort and considerable time for students to internalise such subtle mental schemas as that which lies behind proof, and that this probably has to be done *well before* students encounter the increased intensity which characterises higher level mathematics courses; and
(iii) the central importance of providing a robust template - such as a modified version of Euclid Book I, or the standard sequence of ruler and compass constructions - which students can use as a model for the overall process of 'local deduction plus precedence-respecting logical hierarchy' in mathematics."
University reader in mathematics and mathematics education

"The current school curriculum, particularly up to GCSE but also even at A-level, fails to give pupils much idea of the nature of mathematics as an intellectual subject. ... Euclidean geometry is an ideal topic in this context, since it can be handled at school level and give some idea of the intellectual nature of mathematics."
Chair of Mathematical Physics Group of Institute of Physics, and ex-school teacher

"Engineers and engineering technicians need to be able to use geometry to: lay out patterns on to surfaces for manufacture, plot the trajectory of a robot arm, understand the methods that are used in surveying, e.g. for the siting of antennae."
Head of university department of electronic engineering

"Geometrical and algebraic thinking belong together and support each other; visual stimuli can be a source for algebra, and algebra can be supported by and can inform geometrical thinking. ... Drawing diagrams and appreciating generality implied by a single diagram are non-trivial experiences."
University professor of mathematics education

"I see an increasing use of concrete (as opposed to abstract) algebraic geometry in computer graphics, finite element modelling and data approximation. ... Some relevant introduction at school and university level is highly desirable: topics such as transformation of 3D variables, de Castlejau's algorithm (Bernstein-Bézier polynomials) for shape preserving, fitting piecewise lines to data or curves."
University professor of computational mathematics

"We now teach Analytic Geometry as part of our first year course. Before 1990, calculus was taught without such a preliminary, because students used to come to us with more background in geometry and logical reasoning."
University lecturer in mathematics

"Coordinate geometry should have real geometrical content and not just be a setting for exercises in algebra. ...It is important to develop both visual imagination and logical reasoning."
University pro-vice-chancellor and professor of mathematics

"I do not think that there is any point in trying to learn geometry as a spectator, by which I mean being shown pretty pictures, being told certain geometrical facts, but largely avoiding getting to grips with proofs, reasoning, and problem-solving. ... (Dynamic geometry packages) are

useful tools for the geometer, ... [but] I do not think that playing around with these packages is any substitute for learning how to construct and write out an argument."
University lecturer in mathematics

"Symmetry was one of the themes of modern maths in the late 1960s. There were attempts to develop a secondary school geometry of symmetry, ... but they never cohered enough to be generally adopted because a gradual developmental sequence was not constructed (integrating congruence with symmetry) and few teachers had a background of elementary (and surprising) theorems about symmetry which would have let them match their psychological awareness of pupils with steps forward in the subject. I regularly tried to help student teachers have the background I refer to, and again and again saw them operating splendidly with symmetry in school - better than any textbook."
Professor of mathematics education

"Geometry (in the late 60s) seemed to retreat in favour of the new transformation geometry (followed later by the further de-intellectualisation with the switch to making patterns with flips, slides and turns.) ... I think there are two major things which we lost with the demise of geometry. ... The first important thing is the nature of proof. ... The second important thing is the multi-step solution. The recent style of mathematics examinations has led to children being presented with the solution to problems already planned by the question setter. All the children have to do is to complete the one-step arithmetic manipulations in isolation."
Grammar school head of mathematics

"I would like to see a move away from the culture which regards geometry (and all of mathematics) as an experimental science, in which general truths emerge as mysterious laws of nature. The point of proof at school level is that it provides *explanations*."
University professor of (applied) mathematics

"Euclidean geometry promotes geometric intuition, it is a good place for students to learn about formal proof, and it is a good vehicle for teaching problem solving. These are all obviously important for students who go on to study mathematics at A-level and beyond, because they provide a foundation on which to build. Even more importantly, however, they are very important transferable skills. If they are not included in the curriculum by age 16, many students will never encounter them in as accessible a form and with time to assimilate them properly."
Education Committee of the London Mathematical Society

"[We should] re-instate a much fuller treatment of the conic sections. One of the pinnacles of our subject is the story of the deduction of the planetary orbits from Newton's Laws."
University professor of (applied) mathematics

"I would support moves to give geometry a more significant role, but would be very concerned about the fragmentation of the mathematics curriculum. We seem to be moving towards an American style situation with separate topics whereas the strength of mathematics comes from a unified view."
University professor of mathematics education

"We have noticed that through a practical approach in the study of solid objects and computer simulations, most pupils, including those in special schools, can access the basic elements of study of shape and space. ... It is interesting how easily the geometrical definitions emerge and evolve and abstract structures are formed through this form of investigation as compared to a deductive, formal presentation of properties and definitions offered and confined by the study of the theorems of Euclid."
University professor of mathematics education

"Greater emphasis needs to be placed on teaching Euclidean Geometry in a practical way - taking pupils on maths trails, estimating and measuring the height of buildings, calculating the area of 'odd shaped' rooms for carpets, etc. Pythagoras's theorem should be introduced at level 7 only after practical application. Many builders know all about a 3,4,5 triangle but have no idea they are using Pythagoras's theorem."
Primary school mathematics teacher

"A balance between informal practical experience and formal deductive approaches ... is important at *all* levels. ... The needs of *all* students are important."
University lecturer in education

"Three pervasive problems in the teaching and learning of school geometry are manifest, both in the past and in the present: the separation of the intuitive mathematisation of space from formal definitions and inductive reasoning; the algebraisation of geometry and the suppression of geometric thinking; and the dominance of perception over geometric argument. ... Research into the teaching and learning of Euclidean geometry showed that most students found it hard ... either to follow or to construct proofs in the context of geometry [Freudenthal 1973]. ... Freudenthal suggested that the failure of geometry could be traced to the way geometry was taught in school: 'deductivity was not taught as reinvention, as Socrates did, but was imposed on the learner'."
University professor of education

"There is no disagreement among the mathematics community that proof is a central idea in mathematics and that it is important that all students should meet proof in number of guises within their core 5-16 mathematics curriculum entitlement. Disagreement arises when one particular faction starts to postulate that proof is so central to a particular approach that all will know mathematical proof by being taught a prescribed list of individual content items."
Curriculum Group of the Association of Teachers of Mathematics

"Most present supported the view that there were clear benefits to be gained in the teaching of the formal style of proof incorporated within the more traditional area of geometry. ... However, the firm opinions were shared by all that such an area of mathematics had little relevance to modern students; that it would take up a disproportionate amount of teaching time for the negligible benefits accruing to the few students who might appreciate/enjoy/use such skills at, or beyond, GCSE level; and that there were other more important and relevant topics which could (possibly should) be re-introduced at this level (matrices were mentioned at this stage)."

"We feel that there is enough geometry in GCSE (some would throw out the circle theorems) and A-level. No one wishes to return to the formality of Euclid, which was accessible only to the few and bored, horrified and alienated the many."

"I think it was geometry which persuaded me to become a mathematician. The revelation that the simplest possible rectilinear figure - the triangle - contained such a treasure-trove of unexpected properties - such as the nine-point circle and Euler line - and all through the operation of pure thought, ... struck me, as a thirteen year old, as quite extraordinary."
Mathematics departments and teachers in independent schools

"In my work with designers and other illustrators, I find a basic lack of understanding and the ability to handle simple two dimensional problems."
Technical author and illustrator

"Why does mathematics have much more curriculum time than music? ... Perhaps it is because: Mathematics is more practically useful than music. Have you considered seriously the needs of engineers and physicists, of economists and biologists, of designers and accountants, as well as of citizens? ... Emphasis on formal Euclidean Geometry, rather than the more useful aspects of geometry, is likely to help exacerbate the social divisions that many, including the Government, are trying to reduce."
University professor of mathematics education

"I thought that it was established a long time ago (1868) that Euclidean geometry (the more formal approach à la Euclid) was not the best vehicle for teaching proof, or indeed for teaching geometry!"
Former director of mathematics centre

Appendix 2: National and international contexts for mathematics

1 Introduction

This report has been written in a period of continuing change within the English education system. Recent changes affecting mathematics include the 1999 revision of the National Curriculum, the consequent revision of GCSE syllabuses, the implementation of the National Numeracy Strategy in primary schools, the extension of the pilot National Strategy for Mathematics at Key Stage 3, revisions to the structure of qualifications in 16-19 education (Curriculum 2000) to encourage breadth of study, new specifications for A- and AS-level mathematics and new qualifications in mathematics such as Free Standing Mathematics Units and an AS-level in the Use of Mathematics. These changes are taking place at a time of great difficulty nationally in the recruitment and retention of teachers, especially secondary school teachers of mathematics.

There have been substantial changes in geometry education in the second half of the twentieth century. Two features are worthy of note here. First, the needs of the sciences and engineering have led to a gradually increasing emphasis on the 'applicable' geometrical content embodied in coordinate geometry and vectors (and more recently also in transformations and matrices) at the expense of the 'purer' mathematics of classical 'Euclid'. The position of trigonometry has remained fairly constant. Second, reform in the structure of 11-19 educational institutions and of the examination system has placed a requirement on the mathematics curriculum to be accessible to a wider ability range than was previously the case.

2 11-16 curriculum

The National Curriculum applies to pupils in maintained schools up to the age of 16 and is intended to ensure that pupils are given access to a broad curriculum. Mathematics is one of the core subjects and is compulsory at Key Stage 3 (11-14 age group) and Key Stage 4 (14-16 age group). The Key Stage 4 mathematics curriculum is now divided into two programmes of study - foundation and higher. The higher programme of study is designed for approximately 50% of any age cohort (ie pupils who have attained a secure level 5 at the end of Key Stage 3). The aims and objectives underlying the choice of subject matter are not set out in the Orders; however the handbook on the National Curriculum indicates the skills and experiences pupils are expected to gain from following courses of study based on the National Curriculum. The handbook also requires teachers to provide differentiated programmes of study for those pupils whose attainments fall significantly below or significantly exceed the expected level of attainment. The National Curriculum sets out what is to be taught but not how it is to be taught.

The Qualifications and Curriculum Authority has published schemes of work for many subjects, such as science and ICT, at Key Stage 3. A national Key Stage 3 mathematics strategy is to be introduced in September 2001. That strategy has already led to the publication of a document *Framework for teaching mathematics: Years 7 - 9* which was made available to secondary schools during the Summer Term 2001. This is intended to provide practical support and guidance on teaching mathematics at Key Stage 3. We give some examples of this in Appendix 7. The framework includes yearly teaching programmes, and advice on teaching mathematics lessons and assessing pupils' progress, accompanied by detailed sets of objectives. Associated with the strategy is a programme of professional development for secondary school mathematics teachers which will be supported by a team of some 200 Key Stage 3 mathematics consultants.

The National Curriculum for mathematics at Key Stages 3 and 4 is described under three headings: Ma2 *Number and algebra*, Ma3 *Shape, space and measures* and Ma4 *Handling data*. Together with *Using and applying mathematics*, these also form the attainment targets for assessment purposes. The geometry curriculum is included in the attainment target Ma3*Shape, space and measures* and is divided into four sections:

a) using and applying shape, space and measures covering problem solving, communicating and reasoning;
b) geometrical reasoning covering angles, the properties of rectilinear shapes and circles, and trigonometry, together with a small amount of 3-D work;
c) transformations, coordinates and vectors; and
d) measures and construction (which also includes loci).

There are some key changes in the 1999 revision of the National Curriculum. The topics relating to basic plane geometry are described in more detail. Pupils are now specifically required to solve multi-step problems, and to learn about and create proofs. The degree of sophistication required depends upon age and prior attainment.

The core subjects of the National Curriculum, including mathematics, are assessed at the end of Key Stage 3 by national tests, and at the end of Key Stage 4 through GCSEs. These results are published nationally. There are optional tests available for schools to use at the end of Year 7 and Year 8.

3 16-19 curriculum

Following the end of compulsory education at age 16 there is no requirement to continue to study mathematics. About two thirds of the age cohort of c.600 000 choose to stay in full time education. Some of these students continue their studies in schools, others in the Further Education sector, which includes Sixth Form Colleges. As well as the academic A- and AS-level GCE mathematics courses there is a range of courses leading to other qualifications. Results for schools and colleges in A- and AS-level and other examinations are also published nationally.

The newly revised AS/A-level core for pure mathematics contains a small amount of work in coordinate geometry, vectors and trigonometry. Additionally the material required to be studied in respect of functions makes use of skills learnt in geometry at GCSE. Vectors and aspects of the geometry learnt at GCSE are used further in optional modules on mechanics. There are new qualifications called Free Standing Mathematics Units (FSMUs): each requires 60 hours of study and is assessed by both coursework and examination. Most also specify the use of Information and Communication Technology tools such as graphing calculators, graph plotting software and spreadsheets. These units are available at three levels; units at level 3 are called 'advanced' and are equivalent to modules of an A- or AS-level course. There are two units in geometry but neither is at level 3. A new qualification will be offered for the first time in September 2001. This is AS-level 'Use of mathematics' and comprises two level 3 FSMUs together with a new synoptic unit. The current draft of this qualification has a strong emphasis on mathematical modelling using functions and data, without any geometrical content.

4 Setting and differentiation

Almost all secondary schools teach mathematics up to the end of Key Stage 4 in sets organised by attainment. Consequently the rate at which pupils progress through the 11-16 curriculum depends mainly upon prior attainment and the requirement to provide differentiated programmes of study for those pupils whose attainments fall significantly below or significantly exceed the expected level of attainment. Such differentiation is reflected in the tiered examination structure adopted at GCSE, which has different syllabuses for the foundation, intermediate and higher tiers. There is no such differentiation in A- and AS-level mathematics where all candidates are required to study a common core, which comprises 50% of the content.

5 Examination results

The proportion of pupils achieving five A*-C grades at GCSE has risen steadily to 49% of the cohort (from about 30%) since the introduction of the GCSE in 1988. Around 46% of GCSE candidates in Year 11 (15-16 year olds) currently achieve A*-C grades in mathematics. Currently more than 60 % of those obtaining good (A*-B) grades in GCSE mathematics choose not continue with the subject.

Of the c.400 000 students staying in full time education after 16, some 225 000 enter for one or more A-levels. Mathematics is currently the third most popular subject (behind English and Social Studies). Entries for A-level in 2000 in the 16-18 age group were around 54 000 for Mathematics and about 5 000 for Further Mathematics, with a further 11,500 for AS-level. The pattern of entry changed from September 2000 with the start of Curriculum 2000 where students are encouraged to start by studying four, or perhaps five, subjects. Of those 16-18 year olds completing Mathematics and Further Mathematics A-level in 2000 around 48% achieved a pass at grade A or B, with nearly 29% achieving grade A. At AS-level, 18% gained grade A or B and 9% gained grade A. In 1999 only about 3200 students under the age of 20 with A-level mathematics applied to read mathematics at university in England and Wales. The report '*Measuring the Mathematics Problem*' (Engineering Council et al, 2000) presents evidence of a marked decline in university entrants' mastery of mathematics skills and their level of preparation for mathematics based degree courses. It notes that "*This decline is well established and affects students at all levels*" and also that "*There is an increasing inhomogeneity in the mathematical attainments and knowledge of students entering science and engineering degree programmes*".

6 The Third International Mathematics and Science Study (TIMSS)

England has taken part in two recent large scale international comparisons of mathematics standards under the title of the Third International Mathematics and Science Study (TIMSS). The first report of 1996 gave the results of testing Years 4, 5, 8 and 9 in 1994. The recent repeat testing for Year 9 only in 1998 was reported in December 2000. The overall results in mathematics for Year 9 (mostly 13 year olds) in 1994 showed England as slightly below the international mean, roughly in the same position occupied in every large scale comparison starting in the early 1960s (apparent deviations from this have mainly reflected the composition of countries taking part). In 1994 the English results were either slightly lower than or broadly similar to those of comparable Western European or English-speaking countries. East Asian countries like Singapore, Japan and Korea were significantly ahead of the field, and behind them were several countries in Eastern Europe. Making comparisons with European countries is difficult due to different practices regarding progression from one grade to the next (many countries require students who fail an end of year test or examination to repeat the year) and to the way special educational needs are addressed. The overall position of England was very slightly lower in 1998, after taking into account some differences in the set of countries taking part. However this apparent continuity disguises the fact that boys improved on their 1994 performance, while girls' scores fell. The difference between the sexes in 1998 was the third largest out of 38 countries tested, with only Iran and Tunisia having larger differences.

Uniquely in geometry the English scores deteriorated between 1994 and 1998. In comparison with the other countries taking part, Year 9 students scored just below the average in 1994 and rather more below the average in 1998. In comparison, scores on all other mathematical topics in 1998 were above the international average, and those on Fractions and Number Sense, Algebra and Measurement rose between 1994 and 1998. The low performance in geometry is especially disappointing when set alongside the high scores achieved in geometry by the same cohort of pupils tested in 1994 when they were Year 5. With Hong Kong and Australia, they achieved the highest scores in geometry.

7 International trends in the geometry curriculum

Looking across countries, the TIMSS data show considerable variation in the design and make up of the mathematics curriculum in general and the geometry curriculum in particular [Schmidt 1997]. The same conclusion has been reached by a smaller, but more detailed, study of the geometry curricula of a sample of countries [Hoyles *et al,* 2001]. This study found that few countries have clear mathematical and pedagogical goals with respect to geometry; most offer what appears to be a collection of topics. One country that does illustrate clear goals for geometry is the Netherlands where, in what is described as 'vision geometry', geometry is embedded in practical activities where even 2-D exercises are viewed through a 3-D perspective and there is no mention of proof. Other curricula with a clear focus have a more theoretical orientation, but among these there is evidence of different approaches depending on whether congruence or transformations are used as organising principles and how far proof is to be 'discovered', used or constructed.

Both the study by Hoyles *et al* and the ICMI study on geometry [Mammana & Villani, 1998] indicate that many countries are in the process of changing their geometry curricula with the majority seeking ways to integrate technology. How such integration is to be done is, in most cases, far from clear. Why such integration is happening is clearer, as the following example illustrates. One country that has attempted to teach a theoretical orientated geometry curriculum to *all* pupils is Japan [Howson, 2000]. While recent Japanese research has reported that only about 20% of students are able to solve geometrical proof problems as a result of being taught such a curriculum, a detailed study of students using dynamic geometry software found three major effects: through using such software the students could visualise the geometrical character of a figure more clearly, they had a better understanding of the meaning of the theorem, and were clearer about what they should be proving [Nomura, 1999].

In France, at the time of writing, a major review of the teaching and learning of school mathematics is drawing to a conclusion. In terms of geometry, this review confirms geometry as a vital part of the school mathematics curriculum [Commission de Refléxion sur l'Enseignement des Mathématiques, 2000]. For students aged 11-19 the French report on geometry has five recommendations. These are that geometry for such students should:

a) focus on the understanding of space (by including, for example, the study of polyhedra and of spherical geometry);
b) reinforce the notion of invariance (of length, angle, surface);
c) stress problems concerning geometrical situations and constructions;
d) re-emphasise cases of isometry of triangles; and
e) introduce a rich variety of geometry to students (to include, for example, inversive geometry).

The French report emphasises that the expertise of teachers is crucial to the successful teaching of such a curriculum and recommends the reinforcement of geometry within university curricula for prospective mathematics teachers.

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Appendix 3: Some recent government initiatives in education

1 National Strategies

The government has introduced a number of measures aimed at raising achievement. In primary schools these have been based upon the National Literacy and Numeracy Strategies which are led by the Standards and Effectiveness Unit of the Department for Education and Skills (DfES). These strategies are now supported through framework documents and other materials used extensively in primary schools. It is Key Stage 3 of the National Curriculum in secondary and middle schools which is now the subject of attention. A transforming Key Stage 3 strategy has now been launched. (See e.g. http://www.standards.dfee.gov.uk) There are five main strands in the strategy. Two of these - English and mathematics - are to be implemented nationally from September 2001 following a 1-year pilot phase. Meanwhile science and ICT are being piloted before being adopted nationally in September 2002. The fifth strand, entitled 'Teaching and Learning in the Foundation Subjects' will also be implemented nationally in 2002.

2 Funding for initiatives

In addition to the usual funding arrangements, all maintained schools and LEAs can apply for further funding in support of national initiatives through the DfES's Standards Fund. For example this makes provision for schools to improve their hardware, software, resources and training in connection with the government's educational ICT strategy - known as the National Grid for Learning (NGfL). Additional funds are also made available to a variety of schools and colleges through a number of initiatives. These include the specialist and beacon schools, some of which are affiliated to the Technology Colleges Trust, those in the Education Action Zones and those contributing to the 'Excellence in Cities' initiative.

3 The teaching profession

The government has announced its intention to modernise working practices in both education and health. One aspect of this is to increase the professionalism of teachers. As from 2001 all teachers in the maintained sector are required to register with the General Teaching Council. This, together with the National College for School Leadership, will take over the major responsibilities for policy and coordination of teachers' continuing professional development (CPD) from the Teacher Training Agency. The government has already introduced a new career grade for teachers called the 'Advanced Skills Teacher'. Such teachers normally have a responsibility, known as 'outreach', to carry out development work in other schools. A new pay structure has been introduced with the intention of encouraging good teachers to stay in the classroom rather than seek preferment through management and added administration.

Until the National Numeracy Strategy there had been a period of about eight years in which little subject based professional development took place. The government launched its new strategy for teachers' CPD in March 2001 which has a number of strands, such as bursaries, secondments, learning accounts etc. Within this there is a strong emphasis on increasing teachers' subject knowledge, and associated pedagogy. The new Key Stage 3 mathematics strategy will now become a major channel of funds for CPD into mathematics education, through the employment of local consultants and the involvement of a selected group of leading mathematics teachers.

With the upturn in the economy some seven years ago, courses of initial teacher education have struggled to recruit candidates for shortage subjects such as secondary school mathematics. Until recently teaching was one of the professions within which trainees did not receive some form of pay while working towards qualifications. The government redressed this last year and now pays graduate trainees a small annual sum, with additional sums for shortage subjects such as mathematics. The government is also considering paying off the student loans of some teachers - including mathematics teachers - who enter and stay within the maintained sector for a specified number of years.

The government has introduced a set of standards and a National Curriculum for initial teacher training (ITT) which trainees must satisfy in order to be recommended for Qualified Teacher Status (QTS). The ITT National Curriculum for secondary level mathematics specifies the essential core of knowledge, understanding and skills which all trainees must be taught and be able to use in their teaching. While regrettably containing little reference to geometry, it does include the ability to use ICT effectively in the teaching and learning of the subject.

This ICT specification for trainees has also been embraced as the goal for the subject specific ICT training for teachers in post, organised through the New Opportunities Fund. Here £230m is being spent over four years to improve the skills of around 400 000 teachers and school librarians in the educational use of ICT. Much of this training is taking place in the teachers' own time. To assist in the identification of training needs, the TTA provided schools in 1999 with a set of CD-ROMs containing information and case studies about the use of ICT in subject teaching for needs identification purposes. One of the four secondary mathematics case studies includes a classroom video of a teacher using dynamic geometry software to teach a Year 7 class about parallel lines.

4 ICT and the curriculum

In the 1999 revision of the National Curriculum the descriptions of many subjects were updated in a way which considerably strengthened their references to the educational uses of ICT, but this was not done effectively in the case of mathematics. The new framework document for Key Stage 3 mathematics demonstrates a much more positive view of the potential of ICT to enhance teaching and learning if used critically. Associated with this is a government project to use ICT to enhance teaching and learning of mathematics in Year 7. The DfES has also extended its Computers for Teachers initiative into a second phase aimed specifically at those teaching Key Stage 3 mathematics. This enables qualifying teachers to reclaim up to £500 towards the cost of buying a computer.

Appendix 4: Expectations of geometry in education

1 Geometry for pupils following the foundation programme of study at Key Stage 4

The view of the working group is that the revised National Curriculum forms the basis for creating a rewarding geometry curriculum for these pupils. These pupils are unlikely to study mathematics at a level beyond GCSE. Thus their studies of mathematics during Key Stages 3 and 4 need to provide the foundations for the mathematics which are they are likely to need in pursuing other studies, work and everyday life. It is primarily the role of the teacher to provide a rewarding geometry curriculum in the classroom. The approach to geometry needs to be attainable, demanding, and interesting. For most of these pupils it is necessary to connect work in the mathematics classroom to the world outside in ways that seek to engage their interest. Three principle objectives in teaching geometry to this group are:

a) to build up knowledge and understanding of geometry and geometrical properties – using the properties of plane figures and solids to deduce more facts and to reason geometrically;
b) to use geometry practically and to solve practical problems both inside and outside mathematics, and
c) to develop an understanding of how geometry describes objects and how it can be used to create them.

Examples of how this might be done are discussed briefly in the next two paragraphs.

The geometry curriculum can be used to provide natural opportunities for justification and deduction at a level basic enough for the great majority of students. Using results such as those involving angles and parallel lines to solve problems can give a sound introduction to justification and logical argument. Some pupils will find it easier to explain their reasoning in those problems where they have physical objects and diagrams to manipulate. Numerical or algebraic problems often need greater levels of articulation and communication skills which can hinder many pupils' explanation of the underlying mathematics.

The geometry curriculum should contribute significantly to the development of pupils' spatial reasoning and their ability to visualise, which should help them to understand the world around them. It should contribute to developing a variety of skills for use in their everyday life. For many pupils there will also be pleasure to be had in the design aspects of the subject.

2 Geometry for pupils following the higher tier syllabus at GCSE

Those pupils who follow the higher tier syllabus at GCSE will include a wide range of attainment, potential and mathematical ambition, yet in many schools they will be taught in the same group. Not all will proceed with their study of mathematics after 16; certainly some with mathematical talent will not proceed to AS-level mathematics. It is important therefore that the geometry that these students study in the 11-16 period both forms a foundation for AS- and A-level work and forms a satisfactory basis for the study of other subjects at AS- and A-level, and at degree level (as far as is possible for subjects that do not require a pass in AS- or A-level mathematics), yet also forms a satisfying study in its own right. It is important that the curriculum tempts as many as possible of this group into entering for AS-level mathematics.

There has been concern, which we share, that many of the more able pupils have been insufficiently stimulated by the geometry they have met. The new National Curriculum has established two different programmes of study at Key Stage 4; namely foundation and higher. The higher programme, the one relevant to the more able pupils, includes extra geometrical content, much in basic plane geometry, and puts more emphasis on reasoning and proof. In addition we would wish there to be more emphasis on tackling problems which require unsignalled steps, on deeper investigations either within geometry or in its applications, and on practising and choosing the routine procedures involved in such work. These extra emphases need to be handled carefully, however, if geometry is to engage, intrigue and sustain pupils so that more will seek to continue their study of mathematics.

3 Geometry and Higher Education in mathematics

The reports '*Measuring the Mathematics Problem*' and '*Engaging Mathematics*' highlight several problems which are interconnected. Despite the substantial increase of students in Higher Education over the last 15 years, the absolute number of students studying mathematics has remained relatively steady, but with more students choosing the mathematics of finance and related studies. There is a shortage of suitably qualified applicants for teacher training. There has also been a decline in the entry standards for undergraduates in mathematics. Applicants for degrees are perceived as knowing less mathematics and as being less proficient than 15 years ago. The content covered and the skills level required to obtain a high grade at A level have reduced over that period.

Within the mathematics community in Higher Education there is (and there has been for several years) a desire to see the perceived decline reversed. In particular, there is strong support to increase the knowledge and skills of pupils by:

a) demanding sounder mathematical knowledge, particularly in algebra;
b) improving technical proficiency;
c) developing knowledge and understanding of proof;
d) improving the ability to solve multi-step problems; and
e) improving the ability to write out coherent mathematical arguments and explanations.

Many of the submissions to this group, from both HE mathematicians and some teachers, stressed the importance of geometry, especially Euclidean geometry, as a vehicle for teaching an appreciation of proof and as leading naturally towards higher levels of mathematics. (Indeed several of these writers described this area as the most inspirational part of their own school mathematics education.) We note that the new National Curriculum has added some emphasis in this area; and we recognise that, although proof is indeed a key element in geometry, it is essential that its teaching be handled sensitively.

4 Perspectives from others in Higher Education

With the expansion of higher education, many science, engineering and information systems departments, together with other departments such as business studies and economics, no longer require undergraduates to have A-level mathematics and some require no more than a C grade at GCSE. Consequently, many undergraduates are not properly equipped to deal with the mathematical content of their courses at a time when there is a greater need for mathematical knowledge and skills. Furthermore the demands of many arts and social science degrees require good mathematical knowledge and skills, not only in statistics. Mathematical modelling in the biological and social sciences is also rapidly increasing, and will continue to do so throughout the 21st century, in both quantity and sophistication. Geometry permeates almost all these mathematical models.

Geometry is intimately related to appreciation in art, in architecture, and in all physical crafts. At an appropriate level school geometry, more than either of the other two main branches of school mathematics (algebra and number), shows how we think logically about the space in which we live i.e. the space around the surface of a solid planet. The concept of a shadow illustrates this; for example the shadow cast by a tennis ball on the surface of a floodlit court is an ellipse. Further, the distinctions between 2-dimensional and 3-dimensional space engage the imagination with the concept of higher dimensions.

Much of physics, structural engineering, geology, geography, and chemistry deal with solid shapes. Many of the explanations of phenomena in physical and biological sciences as well as in engineering involve geometry as a paramount part. Many explanations depend mainly on geometry linked with another concept, for example in drug design the size and shape of molecules, or in buckling the concept of slenderness. In others, geometry dominates, eg in crystallography, in the notions of strain, shear and isomerism.

Geometry underpins much of information technology. The World Wide Web relies heavily on computer graphics, as do all manner of user interfaces. It might be argued that it is advances in graphical user interfaces that have enabled relatively untrained users to produce complex and sophisticated results without necessarily understanding how applications (such as desktop publishing) work just as few really understand what goes on under the bonnet of a car. The mass utilisation of personal computers would not have been possible without these advances. Computer aided geometrical design has largely supplanted conventional engineering drafting, particularly where more complex shapes are required. It is widely used in manufacturing, entertainment (animation, virtual reality, computer games), publishing, graphic design and the web.

Personal computers now embody in hardware sophisticated 3-D graphics, rendering systems previously restricted to high-end computers or elaborate software obsolete, and there is no question that three-dimensional graphics will become commonplace. The 3-D rules of perspective are used to rotate an object on screen, so as to be able to see it from all sides, and to move it through an architectural landscape, so as to view it from all angles, both outside and inside.

There are downsides, particularly the lack of understanding of the mathematical principles that underlie the packages, failure to understand the limitation of the packages and how to extend systems. Computer graphics is a subject popular with students in higher education but few are equipped, even with A-level mathematics, to cope with the underlying principles without which systems cannot be developed. Computationally, 3-D geometry is considerably more difficult to implement than 2-D geometry, although mathematically it should be within the grasp of these students. A broad geometrical education including spatial reasoning, an understanding of the basic geometrical properties of various objects, parametric representation of curves and surfaces, transformations, and construction, as well as facility with algebraic manipulation, would provide some of the foundations upon which understanding could be built.

5 Employers' perspectives

Employers, both public and private, have concerns about the mathematics skills of many of their employees. The sophistication of the mathematics used throughout the public and private sectors - for example, in healthcare, in education, and on trading floors in banks - has increased substantially over the last 25 years. Generally the workforce needs better mathematical skills if public accountability is to be properly discharged and skill shortages are to be dealt with (including insufficient appropriately qualified IT specialists and teachers). The Confederation of British Industry and many of its members are concerned that the education of many school leavers and new graduates needs to be of a higher standard if the UK economy is to compete effectively in world markets. There are two areas of concern which are particularly relevant to this report:

a) the low mathematical and literacy skills of many school leavers; and
b) the shortfall in mathematics, science, engineering and technology graduates.

Appendix 5: Geometry in history and society

In this Appendix we describe briefly the nature of geometry, as it was developed by early civilisations, the classical Greek and later cultures, as it impinges on modern everyday life, technology and science, and its current status within mathematics as a whole.

1 Early Geometry

Geometry emerged as an essentially practical activity, but soon became applied in the social, religious and economic development of early civilisations. There are many examples of the measurement of heights and distances used in elementary astronomy and the building of temples and palaces in Indian, Mesopotamian, Egyptian, Central American, and other cultures. The elementary properties of the circle, the square and the 'Pythagoras' relation were well known from ancient times. This practical knowledge spread across the Mediterranean, was adapted by the emerging Greek culture and began to develop as an intellectual activity from around 600BC. The whole 'Greek' period stretches from Thales (c 640 BC) to Diophantos (c 250 AD). However, from about the time of Archimedes (d 212 BC) the principal focus of development moves to North Africa. By the end of this period geometry had been applied in both theoretical and practical astronomy (including trigonometry), mechanics, optics, music, geography, and astrology. The intellectual culture of the time attempted to understand nature by rational argument, and the style of the 'Elements' of Euclid thereafter became the model for virtually the whole of mathematical discourse up to the twentieth century.

2 Geometry as an element in the history of culture

Geometry in its different forms has been an important part of many cultures. The motions of the stars and planets have regulated our lives from ancient times, and man's relation with the heavens is expressed through the orientation and proportion of temples and other structures dating back many centuries. The so-called classical problem of doubling the cube originated in a Hindu religious practice which is still carried out today. The proportions of the human body were used to design Hindu temples, and also appear in the work of the modern architect Le Corbusier, while the well-known 'Golden Ratio' can be found in many examples of buildings from classical times to today. The visual impact of geometric symmetry was used to great effect in Islamic design, Celtic ornamentation, and in Japanese 'Wasan' geometry. Different kinds of symmetry are still a powerful feature in the design of many modern artefacts. The concept of geometric proportion gave rise to the idea of the musical ratios first documented by the Pythagoreans, and many examples of the exploration of these and other ratios can be found in the music of many cultures. A large part of classical Greek mathematics was preserved and further developed by Arab scholars and transmitted by them, together with Jewish and Christian scholars, to Western Europe. Thereafter, the cultural dominance of Western Europe led to the establishment of Euclidean methods as the model not only for mathematics, but also for many other forms of reasoning.

In the early Renaissance artists like Giotto began to move away from purely symbolic representation and to develop practical techniques of more 'realistic' forms of representation. The paintings still had to conform to a number of cultural conventions, and so perspective was only one element (but a quite significant one for mathematics) of the development of Renaissance painting and architecture. Brunelleschi's text on proportion and perspective became the source of ideas which culminated in the projective geometry of Desargues. The spherical trigonometry from Arab astronomy was applied to finding the direction of Mecca, and the making of maps by different projections of the sphere onto a plane. Anamorphic projection was also developed and these techniques were employed in the science of cartography. Furthermore, economic expansion demanded more sophisticated techniques for navigation, thus motivating further development of trigonometry and the invention of logarithms.

For a long time, man has been fascinated with aspects of the infinite. Evidence of this can be found in self-replicating patterns, spirals and other designs based on tessellations, dilations and dissections. While the primary purpose of many of these designs was social and symbolic, they did require some fundamental knowledge of spatial properties. More recently projective and other transformations, together with visual representations of 'non-Euclidean' geometry, have inspired the work of contemporary artists such as Mondrian, Dali, Escher and Bridget Riley. Movements like cubism and structuralism which include the work of Picasso, Braque and Kandinsky, for example, have also in part been inspired by mathematical idea.

It is interesting to observe a relationship between geometry and the philosophy of different cultures, particularly through their models of the universe. The heliocentric theory originating with the Greeks and elaborated by Ptolemy became the dominant world view and was adapted by Christian philosophers as the model for man at the centre of the universe. The Platonic solids through their shapes and ratios were considered to have special mystical properties and appear in many representations of astronomical theories. These were used by Kepler to calculate the distances between the planets. Hermetic philosophy developed similar ideas in the "music of the spheres", and this was a strong theme in some of Newton's 'non-mathematical' writings. Galileo's adaptation of Copernicus' heliocentric theory was based on what he considered to be strong scientific evidence. However, this motivated serious discussions about the nature of scientific versus religious 'truth'. The seventeenth century saw enormous changes in philosophical climate, principally due to the gradual adaptation of the new cosmology, which relied heavily on geometry for its logical support, and the synthesis of algebra and geometry achieved by Descartes and Fermat provided a powerful tool for the analytical justification of Newton's theory. The 'Principia' was presented in geometrical form in order to reach a wider audience, which enabled philosophers like Voltaire, Hobbes and Locke to understand and support its arguments. Cartesian rationalism, supported by the new developments in mathematics, provided a climate for Lagrange's great 'System of the World' and the belief in a mechanistic universe. The general view at this time was that the principles of geometry, confirmed by experience, would continue to be so because God had made it that way. Thus geometry was the most perfect and certain of the sciences. There are many examples where Greek, Arab and later mathematicians questioned the independence of Euclid's fifth postulate, and by the end of the eighteenth century mathematicians like Saccheri, Lambert and others paved the way for the invention of non-Euclidean geometries by Gauss, Lobachevsky and Bolyai. This had two principal effects: the realisation of the independence of geometry, and hence mathematics, from the physical world, and it again raised questions about the nature of truth. During the nineteenth century new geometries were developed by Riemann, Moebius and others, but since understanding these now required considerable technical knowledge, the effect on philosophy at that time was marginal. It was the use of non-Euclidean geometry by Einstein to develop relativistic models of the universe and curved space-time, together with the development of visual representations that brought geometry again to the notice of philosophers, and Relativism became an important component of Twentieth Century philosophy. Our modern society is apparently less confident of its power to predict and control the future of the world, and this may be partly a product of the recognition of chaotic phenomena in the mathematics used to describe it.

3 Geometry in the everyday world

Geometry is relevant both inside and outside employment. Workers in many sectors of employment need spatial reasoning together with geometrical understanding to a greater or lesser extent. These include workers in the construction industry, such as scaffolders, bricklayers and joiners, as well as decorators, interior designers, landscape gardeners, planners and architects. Within the manufacturing industry, assembly workers, mechanics, engineers and industrial designers also need such skills.

In everyday life, a wide range of activities is linked to some extent with geometry. Many familiar tasks require spatial reasoning. These include laying carpet in a room, packing cases in a car boot, playing sport or choosing a quick route, whether walking from home to school, driving from Glasgow to Inverness, or sailing across the Atlantic. Likewise many 'Do it Yourself' construction projects around the home and garden involve simple geometrical considerations. Many children, and some adults, play 3-D computer games which, almost by definition, require spatial reasoning and an ability to visualise to play them successfully.

4 Geometry in technology

Each of the artefacts of our technological society possesses a shape which is important to its correct functioning, and which must be designed by engineers or architects. The wheel, embodying rotational symmetry, is perhaps the most ancient example. Static frameworks such as the Eiffel Tower, Sydney Harbour Bridge and Buckminster Fuller's geodesic domes are geometry writ large, as are structures based on curved surfaces such as the Members' Stand at the Lords cricket ground, the Sydney Opera House and the Millennium Dome. Geometrical principles underlie the working of all mechanisms and machines, from old steam locomotives to modern robots, through levers, linkages and gears to the Wankel engine. Geometry underpins the successful aerodynamic design of economical cars, trophy winning racing yachts and wingless space shuttles. Computer software has been designed to implement geometrical principles in tackling tasks as diverse as 3-dimensional design, virtual reality and remote controlled robotic surgery. Medicine is also reliant on the 3-D awareness of its surgeons in performing complex operations, and of its radiologists in shaping intersecting beams of radiation to destroy tumours.

5 Geometry in the sciences

The 'purer' sciences are also pervaded by geometrical aspects. From the molecular to the tectonic level, shape, together with physical and chemical properties, governs the natural world. The Human Genome Project has sequenced our DNA, but finding the 3-D configuration of the proteins it specifies is much more difficult, yet essential for the rational design of advanced drugs. The design of advanced materials such as high temperature superconductors, strong light composites, efficient catalysts or fast semiconductors involves a geometrical understanding of their atomic structure. Computer based imaging of 3-D information resulting from a medical scan, a seismic survey, or a mathematical model of stresses, heat or fluid flow, is vital to interpretation. It is also an important aid to the interpretation of multi-dimensional statistical data. Geometry rules the deepest levels of theoretical physics: on the largest scale Einstein's General Theory of Relativity is essentially a geometrical description of space-time, while on the smallest scale, the subtle symmetries and extra dimensions of field theories are vital to elementary particle physics.

6 Geometry in modern mathematics

The place of geometry in mathematics, like mathematics itself, has evolved substantially, and continues to do so. Developments in astronomy led to the invention of trigonometry and spherical geometry, which in their turn required better techniques for computation. The analytic and projective geometry of the seventeenth century brought new techniques into mathematics, and the geometrical principles involved are still fundamental to calculus and its applications today. By the early twentieth century the invention of vectors and matrices had provided new tools whereby algebraic and vector geometries could be developed.

In the nineteenth and early twentieth century projective geometry was commonly the cornerstone of many university courses, and well into the nineteenth century the term 'Geometrician' was still used in continental Europe to describe an expert mathematician. However, new problems and new interests arise, and the study of non-Euclidean geometries, although seemingly very abstract, have since been introduced which are central to modern views of the geometry of space-time. Both algebraic geometry and the analytic study of curves and surfaces, differential geometry, have undergone substantial development throughout the twentieth century.

At the other end of the spectrum from Euclidean geometry is the study of topology or 'rubber sheet geometry', which arises both from Poincaré's work on dynamical systems, and Frechet's desire to unify Cantor's theory of point sets and the treatment of functions as points of a space. Other modern mathematical topics which link geometry with dynamics and analysis include chaos theory, catastrophe theory and fractal geometry.

7 Geometry in future mathematical research

Geometry lies at the heart of future research in the whole of mathematics, both pure and applied. In particular the following geometrical subjects are currently flourishing in research: algebraic geometry, differential geometry, symplectic geometry, computational geometry, topology, algebraic topology, manifold theory, singularity theory and graph theory. Geometry is also fundamental to the future of other mainstream branches of pure mathematics such as group theory, Lie groups, representation theory, commutative rings, algebraic number theory, qualitative differential equations, dynamical systems, chaos and global analysis.

Geometry is fundamental to theoretical physics in relativity, quantum mechanics, field theory, electromagnetism and gravity, and also figures largely in many other areas in which mathematics is applied, such as fluid mechanics, geophysics, defectology, robotics, computer imaging, computer vision, computer aided design, molecular analysis, medical imaging, neural networks, theoretical biology, and modelling in the social sciences.

The importance of geometry in modern mathematics can be seen in the 6 month long research programmes run by the Newton Institute in Cambridge since its foundation in 1992: of these 50% have involved geometry either as the main component or as a major component in both pure and applied programmes. Another example is the policy of the National Science Foundation in the USA: during the last decade, amongst grants in mathematics, priority has been given to those containing a major component of geometry.

Two UK geometers have been awarded the Fields Medal (the highest international award in mathematics, equivalent to a Nobel Prize): Sir Michael Atiyah and Professor Simon Donaldson. At present the UK is one of the leading countries in the world in mathematical research, but if we are to retain this enviable position it is essential to train future generations at school in the foundation of mainstream mathematics, of which geometry has always been, and still is, a major part.

Appendix 6: Geometry in the current 11-16 curriculum

The 1999 revision of '*Mathematics: The National Curriculum for England*' can be downloaded from http://www.nc.uk.net/ Set out below is a summary of the geometry content of the Key Stage 3 and Key Stage 4 mathematics curriculum. Topics in the higher programme of study are given in italics.

1 Geometrical reasoning

Pupils should be taught to:

Angles

a) recall and use properties of angles at a point, angles on a straight line (including right angles), perpendicular lines, and opposite angles at a vertex;
b) distinguish between acute, obtuse, reflex and right angles; estimate the size of an angle in degrees;

Properties of triangles and other rectilinear shapes

c) *distinguish between lines and line segments;* use parallel lines, alternate angles and corresponding angles; understand the properties of parallelograms and a proof that the angle sum of a triangle is 180 degrees; understand a proof that the exterior angle of a triangle is equal to the sum of the interior angles at the other two vertices;
d) use angle properties of equilateral, isosceles and right-angled triangles; understand congruence, recognising when two triangles are congruent; explain why the angle sum of any quadrilateral is 360 degrees;
e) use their knowledge of rectangles, parallelograms and triangles to deduce formulae for the area of a parallelogram, and a triangle, from the formula for the area of a rectangle;
f) recall the essential properties of special types of quadrilateral, including square, rectangle, parallelogram, trapezium and rhombus; classify quadrilaterals by their geometric properties;
g) calculate and use the sums of the interior and exterior angles of quadrilaterals, pentagons and hexagons; calculate and use the angles of regular polygons;
h) *understand and use SSS, SAS, ASA and RHS conditions to prove the congruence of triangles using formal arguments, and to verify standard ruler and compass constructions;*
i) understand, recall and use Pythagoras's theorem *in 2-D, then 3-D problems; investigate the geometry of cuboids including cubes, and shapes made from cuboids, including the use of Pythagoras's theorem to calculate lengths in three dimensions;*
j) *understand similarity of triangles and of other plane figures, and use this to make geometric inferences; understand, recall and use trigonometrical relationships in right-angled triangles, and use these to solve problems, including those involving bearings, then use these relationships in 3-D contexts, including finding the angles between a line and a plane (but not the angle between two planes or between two skew lines); calculate the area of a triangle using $\frac{1}{2}$absinC; draw, sketch and describe the graphs of trigonometric functions for angles of any size, including transformations involving scalings in either or both the x and y directions; use the sine and cosine rules to solve 2-D and 3-D problems.*

Properties of circles

k) recall the definition of a circle and the meaning of related terms, including centre, radius, chord, diameter, circumference, tangent, arc, sector and segment; understand that the tangent at any point on a circle is perpendicular to the radius at that point; *understand and use the fact that tangents from an external point are equal in length;* explain why the perpendicular from the centre to a chord bisects the chord; understand that inscribed regular polygons can be constructed by equal division of a circle; *prove and use the facts that the angle subtended by an arc at the centre of a circle is twice the angle subtended at any point on the circumference, the angle subtended at the circumference by a semicircle is a right angle, that angles in the same segment are equal, and that opposite angles of a cyclic quadrilateral sum to 180 degrees; prove and use the alternate segment theorem*

3-D shapes

l) explore the geometry of cuboids (including cubes), and shapes made from cuboids
m) use 2-D representations of 3-D shapes and analyse 3-D shapes through 2-D projections and cross-sections, including plan and elevation; *solve problems involving surface areas and volumes of prisms, pyramids, cylinders, cones and spheres; solve problems involving more complex shapes and solids, including segments of circles and frustums of cones.*

2 Transformations and coordinates

Pupils should be taught to:

Specifying transformations

a) understand that rotations are specified by a centre and an (anticlockwise) angle; *use any point as the centre of rotation;* use right angles, fractions of a turn or degrees to measure the angle of rotation; understand that reflections are specified by a mirror line, translations by a distance and direction *(or a vector),* and enlargements by a centre and positive scale factor;

Properties of transformations

b) recognise and visualise rotations, reflections and translations, including reflection symmetry of 2-D and 3-D shapes, and rotation symmetry of 2-D shapes; transform 2-D shapes by translation, rotation and reflection, *and combinations of these transformations; use congruence to show that translations, rotations and reflections preserve length and angle, so that any figure is congruent to its image under any of these transformations; distinguish properties that are preserved under particular transformations* recognising that these transformations preserve length and angle, so that any figure is congruent to its image under any of these transformations;

c) recognise, visualise and construct enlargements of objects using positive integer scale factors greater than one, then positive scale factors less than one; understand from this that any two circles and any two squares are mathematically similar, while, in general, two rectangles are not, *then use positive fractional and negative scale factors;*

d) recognise that enlargements preserve angle but not length; identify the scale factor of an enlargement as the ratio of the lengths of any two corresponding line segments and apply this to triangles; understand the implications of enlargement for perimeter; use and interpret maps and scale drawings; understand the implications of enlargement for area and for volume; distinguish between *(understand)* formulae for perimeter, area and volume by considering dimensions; understand and use simple examples of the relationship between enlargement and areas and volumes of shapes and solids; *understand and use the effect of enlargement on areas and volumes of shapes and solids;*

Coordinates

e) understand that one coordinate identifies a point on a number line, two coordinates identify a point in a plane and three coordinates identify a point in space, using the terms '1-D', '2-D' and '3-D'; use axes and coordinates to specify points in all four quadrants; locate points with given coordinates; find the coordinates of points identified by geometrical information [for example, find the coordinates of the fourth vertex of a parallelogram with vertices at (2, 1) (-7, 3) and (5, 6)]; find the coordinates of the midpoint of the line segment AB, given points A and B, then calculate the length AB

Vectors

f) *understand and use vector notation; calculate, and represent graphically the sum of two vectors, the difference of two vectors and a scalar multiple of a vector; calculate the resultant of two vectors; understand and use the commutative and associative properties of vector addition; solve simple geometrical problems in 2-D using vector methods*.

3 Measures and construction

Pupils should be taught to:

Measures

a) interpret scales on a range of measuring instruments, including those for time and mass; know that measurements using real numbers depend on the choice of unit; recognise that measurements given to the nearest whole unit may be inaccurate by up to one half in either direction; convert measurements from one unit to another; know rough metric equivalents of pounds, feet, miles, pints and gallons; make sensible estimates of a range of measures in everyday settings;

b) understand/*use* angle measure, using the associated language [for example, use bearings to specify direction];

c) understand and use compound measures, including speed and density;

Construction

d) measure and draw lines to the nearest millimetre, and angles to the nearest degree; draw triangles and other 2-D shapes using a ruler and protractor, given information about their side lengths and angles; understand, from their experience of constructing them, that triangles satisfying SSS, SAS, ASA and RHS are unique, but SSA triangles are not; construct cubes, regular tetrahedra, square-based pyramids and other 3-D shapes from given information;

e) use straight edge and compasses to do standard constructions, including an equilateral triangle with a given side, the midpoint and perpendicular bisector of a line segment, the perpendicular from a point to a line, the perpendicular from a point on a line, and the bisector of an angle;

Mensuration

f) find areas of rectangles, recalling the formula, understanding the connection to counting squares and how it extends this approach; recall and use the formulae for the area of a parallelogram and a triangle; find the surface area of simple shapes using the area formulae for triangles and rectangles; calculate perimeters and areas of shapes made from triangles and rectangles;

g) find volumes of cuboids, recalling the formula and understanding the connection to counting cubes and how it extends this approach; calculate volumes of right prisms and of shapes made from cubes and cuboids;

h) find circumferences of circles and areas enclosed by circles, recalling relevant formulae, *calculate the lengths of arcs and the areas of sectors of circles;*

i) convert between area measures, including square centimetres and square metres, and volume measures, including cubic centimetres and cubic metres;

Loci

j) find loci, both by reasoning and by using ICT to produce shapes and paths [for example, equilateral triangles] *[for example, a region bounded by a circle and an intersecting line]*.

Appendix 7: Geometry in the Key Stage 3 mathematics strategy

The following are extracts from the Key Stage strategy document '*Framework for teaching mathematics: Years 7-9*' which can be downloaded from the Standards website at: http://www.standards.dfee.gov.uk/keystage3/strands/mathematics/

1 Shape, space and measures

Geometry in Key Stage 3 is the study of points, lines and planes and the shapes that they can make, together with a study of plane transformations. A key aspect is the use and development of deductive reasoning in geometric contexts. Geometrical activities can be linked to accurate drawing, construction and loci, and work on measures and mensuration. By ensuring that pupils have a range of suitable experiences you can develop their knowledge and understanding of shape and space and their appreciation of the ways that properties of shapes enrich our culture and environment.

2 Geometrical reasoning

Pupils can be aware of and use geometrical facts or properties that they have discovered intuitively from practical work before they can prove them analytically. The aim in Key Stage 3 is for pupils to use and develop their knowledge of shapes and space to support geometrical reasoning. For example, they need to appreciate that tearing the corners off a triangle and placing them side by side at best indicates that the angle sum of a triangle is approximately 180°, and that however many particular cases they can find of triangles with an angle sum of 180°, this does not prove the general case. In Key Stage 3, you can build on pupils' experience and the practical demonstrations and explanations that have sufficed in Key Stages 1 and 2. Teach them to understand and use short chains of deductive reasoning and results about alternate and corresponding angles to reach a proof. Later, pupils should be able to explain why the angle sum of any quadrilateral is 360°, and to deduce formulae for the area of a parallelogram and of a triangle from the formula for the area of a rectangle. These chains of reasoning are essential steps towards the proofs that are introduced in Key Stage 4.

3 Appreciation of shape and space

Geometry cannot be learned successfully solely as a series of logical results. Pupils also need opportunities to use instruments accurately, draw shapes and appreciate how they can link together, for example, in tessellations. In Key Stage 3, it is vital to distinguish between the imprecision of constructions which involve protractors and rulers, and the 'exactness in principle' of standard constructions which use only compasses and a straight edge. Geometrical reasoning can show pupils why construction methods work, for example, the method to construct a perpendicular bisector of a line segment.

Practical work with transformations will produce interesting problems to solve as well as helping pupils to understand the topic more fully. Urge them to visualise solutions to problems such as: 'When a triangle is rotated through 180 degrees about the mid-point of one side, what shape do the original and final triangles form?' Linking geometry to subjects such as art, through symmetry or tessellations, or religious education, perhaps through a study of the properties of Islamic patterns or cathedral rose windows, offers good opportunities to develop creativity. By encouraging pupils to speculate why the properties they have found hold true you can strengthen their reasoning skills.

4 Use of ICT

ICT offers good opportunities to develop geometrical reasoning and an appreciation of shape and space. For example, pupils can use the programming language Logo to explore properties of plane shapes, such as the exterior angles of polygons. With dynamic geometry software, they can use rapid geometric drawing to explore a condition such as 'one pair of opposite angles of a quadrilateral is equal', and discover the special circumstances under which the condition is true. More able pupils may be able to prove their conjectures analytically, but the formal use of congruent triangles is often needed, and for most pupils this will be tackled in Key Stage 4.

5 Measures and mensuration

Pupils in Key Stage 3 need to develop their awareness of the relative sizes of units, converting between them, and using the rough equivalence of common imperial and metric units. Towards the end of the key stage, they need to become familiar with compound measures such as speed or density. Help them to appreciate the imprecision of measurement and to recognise the accuracy to which measurements can be stated. Draw as far as possible on their practical experience of measures in other subjects, particularly design and technology, science, geography and physical education.

Work on perimeter, area and volume will extend to a range of shapes, including rectangles, parallelograms, circles, cuboids and prisms. A project such as 'design a swimming pool' allows pupils to exercise imagination, practise calculations of length, area and volume, and experience working with larger numbers and units. The heart of work on mensuration will be with triangles which, at the end of the key stage, will extend to Pythagoras's theorem and similarity, leading on to trigonometry. As far as possible, the relevant formulae for calculating perimeters, areas and volumes should be explored and justified logically, not simply stated as facts. The use of formulae can be linked to work in algebra, and can be enhanced by the use of spreadsheets and graphical calculators.

6 Features of shape, space and measures in Key Stage 3

To summarise, the distinctive features of shape, space and measures in Key Stage 3 are:

- developing geometrical reasoning and construction skills, and an appreciation of logical deduction;
- developing visualisation and sketching skills, including a dynamic approach to geometry, making use of ICT and other visual aids;
- developing awareness of the degree of accuracy of measurements.

7 Key Objectives – geometry only

Year 7

- Plot the graphs of simple linear functions.
- Identify parallel and perpendicular lines; know the sum of angles at a point, on a straight line and in a triangle.

Year 8

- Enlarge 2-D shapes, given a centre of enlargement and a positive whole-number scale factor.
- Use straight edge and compasses to do standard constructions.
- Deduce and use formulae for the area of a triangle and parallelogram, and the volume of a cuboid; calculate volumes and surface areas of cuboids.
- Identify the necessary information to solve a problem; represent problems and interpret solutions in algebraic, geometric or graphical form.
- Use logical argument to establish the truth of a statement.

Year 9

- Construct functions arising from real-life problems and plot their corresponding graphs; interpret graphs arising from real situations.
- Solve geometrical problems using properties of angles, of parallel and intersecting lines, and of triangles and other polygons.
- Know that translations, rotations and reflections preserve length and angle and map objects on to congruent images.
- Know and use the formulae for the circumference and area of a circle.
- Present a concise, reasoned argument, using symbols, diagrams, graphs and related explanatory text.

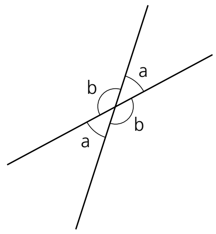
Year 9 objectives for able pupils

- Know that if two 2-D shapes are similar, corresponding angles are equal and corresponding sides are in the same ratio.
- Understand and apply Pythagoras' theorem.
- Know from experience of constructing them that triangles given SSS, SAS, ASA or RHS are unique, but that triangles given SSA or AAA are not; apply these conditions to establish the congruence of triangles.

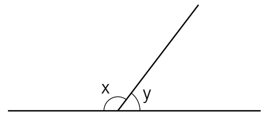
Extracts from '*Supplement of examples: Years 7, 8, and 9*' are given on the following four pages.

Shape, space and measures

Pupils should be taught to:	**As outcomes, Year 7 pupils should, for example:**
Identify properties of angles and parallel and perpendicular lines, and use these properties to solve problems (continued)	Know the sum of angles at a point, on a straight line and in a triangle, and recognise vertically opposite angles and angles on a straight line.



vertically opposite angles



$x + y = 180^\circ$

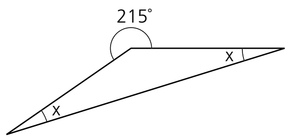
angles on a straight line

Link with rotation.

Recognise from practical work such as measuring and paper folding that the three angles of a triangle add up to 180˚.

Given sufficient information, calculate:
- angles in a straight line and at a point;
- the third angle of a triangle;
- the base angles of an isosceles triangle.

For example:
- Calculate the angles marked by letters.

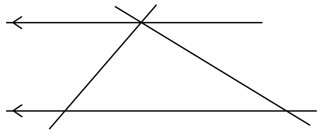


Geometrical reasoning: lines, angles and shapes

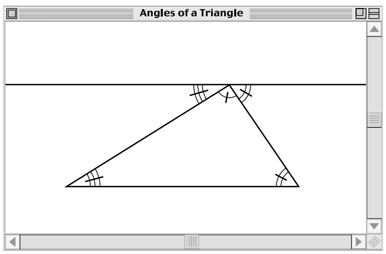
As outcomes, Year 8 pupils should, for example:

Understand a proof that the sum of the angles of a triangle is 180° and of a quadrilateral is 360°, and that the exterior angle of a triangle equals the sum of the two interior opposite angles.

Consider relationships between three lines meeting at a point and a fourth line parallel to one of them.



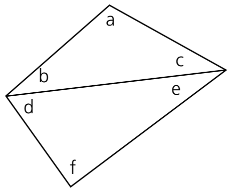
Use dynamic geometry software to construct a triangle with a line through one vertex parallel to the opposite side. Observe the angles as the triangle is changed by dragging any of its vertices.



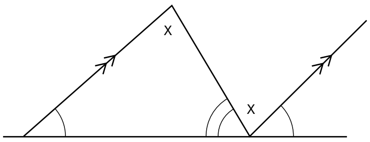
Use this construction, or a similar one, to explain using diagrams a proof that the sum of the three angles of a triangle is 180°.

Use the angle sum of a triangle to prove that the angle sum of a quadrilateral is 360°.

$(a + b + c) + (d + e + f)$
$= 180° + 180° = 360°$

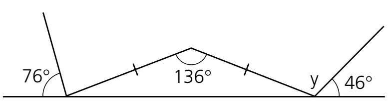


Explain a proof that the exterior angle of a triangle equals the sum of the two interior opposite angles, using this or another construction.



Given sufficient information, calculate:
- interior and exterior angles of triangles;
- interior angles of quadrilaterals.

For example:
- Calculate the angles marked by letters.

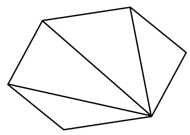


As outcomes, Year 9 pupils should, for example:

Explain how to find, calculate and use properties of the interior and exterior angles of regular and irregular polygons.

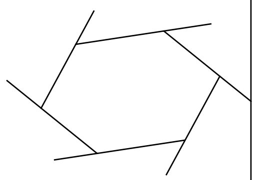
Explain how to find the interior angle sum and the exterior angle sum in (irregular) quadrilaterals, pentagons and hexagons. For example:

- A polygon with n sides can be split into $n - 2$ triangles, each with an angle sum of 180°.



  So the interior angle sum is $(n - 2) \times 180°$, giving 360° for a quadrilateral, 540° for a pentagon and 720° for a hexagon.

  At each vertex, the sum of the interior and exterior angles is 180°.

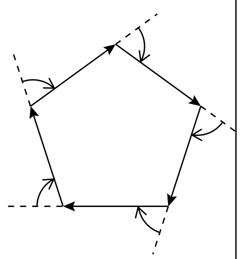


  For n vertices, the sum of n interior and n exterior angles is $n \times 180°$.
  But the sum of the interior angles is $(n–2) \times 180°$, so the sum of the exterior angles is always $2 \times 180° = 360°$.

Find, calculate and use the interior and exterior angles of a regular polygon with n sides.
For example:

- The interior angle sum S for a polygon with n sides is $S = (n - 2) \times 180°$.
  In a regular polygon all the angles are equal, so each interior angle equals S divided by n.
  Since the interior and exterior angles are on a straight line, the exterior angle can be found by subtracting the interior angle from 180°.

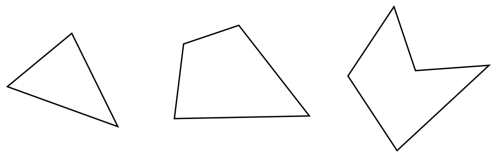
- From experience of using Logo, explain how a complete traverse of the sides of a polygon involves a total turn of 360° and why this is equal to the sum of the exterior angles.



  Deduce interior angle properties from this result.

Recall that the interior angles of an equilateral triangle, a square and a regular hexagon are 60°, 90° and 120° respectively.

Shape, space and measures

Pupils should be taught to:	As outcomes, Year 7 pupils should, for example:
Construct lines, angles and shapes	Use, read and write, spelling correctly: *construct, draw, sketch, measure… perpendicular, distance…* *ruler, protractor (angle measurer), set square…* Use ruler and protractor to measure and draw lines to the nearest millimetre and angles, including reflex angles, to the nearest degree. For example: • Measure the sides and interior angles of these shapes.  See Y456 examples. Link to angle measure.

Construction and loci

As outcomes, Year 8 pupils should, for example:

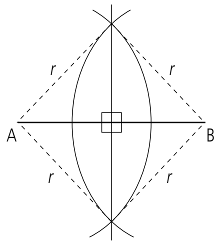
Use vocabulary from previous year and extend to:
bisect, bisector, mid-point… equidistant…
straight edge, compasses… locus, loci …

In work on construction and loci, know that the shortest distance from point P to a given line is taken as the distance from P to the *nearest* point N on the line, so that PN is perpendicular to the given line.

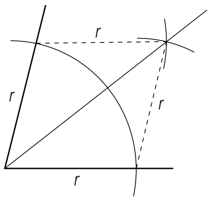
Use straight edge and compasses for constructions.

Recall that the diagonals of a rhombus bisect each other at right angles and also bisect the angles of the rhombus. Recognise how these properties, and the properties of isosceles triangles, are used in standard constructions.

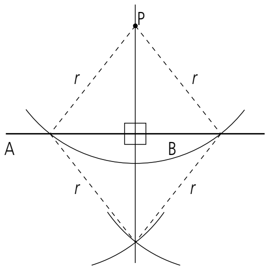
- Construct the mid-point and perpendicular bisector of a line segment AB.



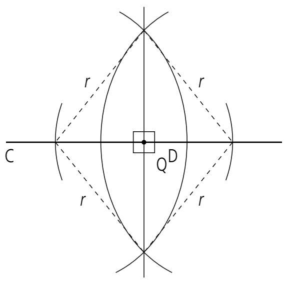
- Construct the bisector of an angle.



- Construct the perpendicular from a point P to a line segment AB.



- Construct the perpendicular from a point Q on a line segment CD.



Know that:
- The perpendicular bisector of a line segment is the locus of points that are equidistant from the two end points of the line segment.
- The bisector of an angle is the locus of points that are equidistant from the two lines.

Link to loci and properties of a rhombus, and to work in design and technology.

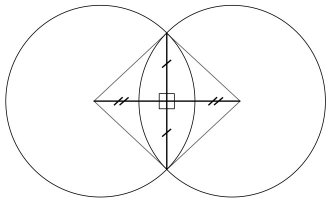
As outcomes, Year 9 pupils should, for example:

Use vocabulary from previous years and extend to:
circumcircle, circumcentre, inscribed circle …

Use straight edge and compasses for constructions.

Understand how standard constructions using straight edge and compasses relate to the properties of two intersecting circles with equal radii:

- The common chord and the line joining the two centres bisect each other at right angles.
- The radii joining the centres to the points of intersection form two isosceles triangles or a rhombus.

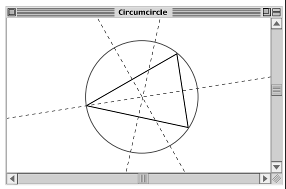


Use congruence to prove that the standard constructions are exact.

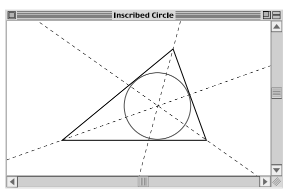
Use construction methods to investigate what happens to the angle bisectors of any triangle, or the perpendicular bisectors of the sides. For example:

- *Observe the position of the centres of these circles as the vertices of the triangles are moved.*

Construct a triangle and the perpendicular bisectors of the sides. Draw the circumcircle.



Construct a triangle and the angle bisectors. Draw the inscribed circle.



Link to properties of a circle, and to work in design and technology.

Appendix 8: Spatial thinking and visualisation

contributed by Keith Jones

1 Spatial Thinking

Among the many modalities of human thought, two are particularly common: verbal reasoning and spatial reasoning. Verbal reasoning is the process of forming ideas by assembling symbols into meaningful sequences. Spatial reasoning is the process of forming ideas through the spatial relationships between objects. It is the form of mental activity which makes it possible to create spatial images and manipulate them in the course of solving practical and theoretical problems. Because space is a fundamental feature of the human environment, spatial thinking plays a crucial role in even the most ordinary human problem solving. People process spatial information when they navigate, when they manipulate objects, and when they design them. Geometry is an example of spatial reasoning at work.

The mathematician Jacques Hadamard argued that much of the thinking that is required in higher mathematics is spatial in nature. Einstein's comments on thinking in images are well known. Numerous mathematicians report using spatial skills when they visualise mathematical relations. Physical scientists also report using such skills when they visualise and reason about the models of the physical world.

Spatial thinking is an important component in solving many types of mathematics problems. This includes the use of diagrams and drawings, searching for patterns and structures, graphing numbers, considering how fractions can be broken down into geometrical regions, conceptualising mathematical functions, and so on. Spatial thinking has an important role in mathematics achievement, with positive correlations found between spatial ability and mathematics achievement at all levels.

Investigative tasks in geometry and measurement provide opportunities for students to analyse mathematically their spatial environment, to describe characteristics and relationships of geometric objects, and to use number concepts in a geometric context. In this way, students develop and use spatial thinking.

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2 Visualisation

Visualisation is generally taken to refer to "*the ability to represent, transform, generate, communicate, document, and reflect on visual information*" [Hershkowitz, 1989]. As such, it is a crucial component of learning geometrical concepts. Moreover, a visual image, by virtue of its concreteness, is "*an essential factor for creating the feeling of self-evidence and immediacy*" [Fischbein, 1987, p.101]. Therefore, it "*not only organizes data at hand in meaningful structures, but it is also an important factor guiding the analytical development of a solution*" [ibid].

Visualisation is essential to problem solving and spatial reasoning as it enables people to use concrete means to grapple with abstract images. In mathematics the process of visualisation entails the process of forming and manipulating images, whether with paper and pencil, technology or mentally, to investigate, discover and understand. The original meaning of the Greek word for 'to prove' (*deiknumi*) was to make visible or show.

There are serious reasons for being good at visualisation. From 2-dimensional pictures, it is often useful to determine the possible shapes of 3-dimensional objects, and vice versa. For instance, doctors and dentists and others in the health profession often need to determine from X-rays or MRI pictures the precise position and shape of an organ or bone or tooth or tumour. Geometry provides the concepts that assist in this work - concepts like cross section and contour curve.

Mathematics has a long tradition of interest in visualisation methods. Such classic works as *Anschauliche Geometrie* by Hilbert and Cohn-Vossen (translated as *Geometry and the Imagination*) demonstrate the influence of this visual approach to mathematics (indeed, it is worth noting that the English translation of the title only barely does justice to the complex nuances of the German *anschaulich*).

While visualisation process has been a cornerstone of the mathematical reasoning process since the times of the ancient geometers, the advent of high-performance interactive computer graphics systems has opened a new era that is still evolving. Mathematical visualisation is about much more than 'pretty' graphics; it has become a mathematical discipline and involves concepts in mathematics with growing implications and applications across a range of disciplines. The aim of mathematical visualisation is to offer efficient visualisation tools for many areas of mathematics, thereby creating tangible experiences of abstract mathematical objects and concepts. Typical geometric problems of interest to mathematical visualisation applications involve both static structures, such as real or complex manifolds, and changing structures requiring animation. In practice, the emphasis is on manifolds of dimension two or three embedded in three or four-dimensional spaces due to the practical limitations of holistic human spatial perception.

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Appendix 9: Proof – 'why and what?'

The main body of this report speaks positively about the role of proof in the teaching and learning of geometry. In this appendix we will try to make clear what we mean by proof, what its role is in school mathematics, and why it is important, and will then give some varied examples of interest.

1 Terminology

First, then, we will try to make clear what we mean by proof. Many near synonyms of the word 'prove', such as 'explain', 'justify', 'deduce', 'demonstrate' 'use reasoning to show', appear in the National Curriculum. We wish to draw a distinction here between some related notions.

a) Firstly, there is a logical argument which demonstrates the truth of some claim - this may be a formal argument or an informal but valid argument which, with skill and experience, could be refined to a formal argument. The words listed above all represent shades of this notion. We term them all as proof here. Proof is a concept which is central to mathematics. By this process, mathematicians over the centuries have built up a huge body of knowledge which is established in a sense which no scientific theory can ever be.
b) Next, there is a notion which arises in relation to a mathematical conjecture and is analogous with what happens with scientific theories. A conjecture, like a theory, will fit known facts and may lead to predictions which can then be checked in specific instances, possibly in very many instances. This we will describe as 'providing evidence' for the conjecture. This evidence provides some reassurance that the conjecture is reasonable or possibly likely to be true. But it does not actually demonstrate its truth; conjectures, like scientific theories, from time to time need to be revised to fit with newer observations or even to be rejected.
c) Finally, it is important to distinguish the mathematical usage of the term proof from that used in everyday language when a strong argument supported by evidence (eg in a court of law or a tribunal) may occasionally be termed a proof. A version of this might appear in the mathematics classrooms when a pupil explains why she or he believes that something is true but without actually providing any logical argument.

A 'theorem' is simply a mathematical statement of some interest which is known to be true because it has a proof, ie a logical demonstration of the truth of that statement. That statement might also be termed a fact or result. A theorem will typically state that, under specific conditions, explicit or implicit, a certain conclusion is true. Its proof will make use of previously established results and logical argument to demonstrate this truth. The process of deduction must be robust, so that if the conditions are satisfied then the facts established by the theorem must be true.

The process of proof must begin somewhere. The starting point for abstract mathematics is a minimal collection of initial reasonable assumptions termed axioms. These then form an implicit part of the assumptions of the results which follow. In the context of school mathematics, however, experience has shown that this is not a sensible approach. Rather, one should start with some well-known or 'obvious' facts which need to be carefully chosen and, in a sense, explicit. Then, using deductive reasoning, a collection of related results, of a less obvious nature, should be built up.

2 Benefits of training in proof

It is true that much that is mathematical can be done without being concerned with proof. Indeed all mathematics prior to the ancient Greeks, and much thereafter, was based on techniques which were seen to work; and one can still simply teach and use mathematical techniques and facts with no mention of proof. Moreover, understanding and producing proofs is not the easiest of skills to teach or learn. So there need to be strong reasons for building proof into the mathematics curriculum.

We note first that it is widely believed that pupils retain mathematics best when they understand it; so it is eminently practical to try to ensure that pupils have such understanding. Now a full understanding of why some fact is true or why some technique works is, essentially, a proof of it. (This, of course, presupposes that the pupil gains such understanding - so the way in which the pupil meets proofs needs great care on the part of the teacher. The results and proofs used need to be matched to the pupil.)

If well done, justifying through proof has a liberating effect on pupils since it enables them to see why results in mathematics are true and why particular mathematical techniques work. They no longer need to accept this on the authority of the teacher or textbook - instead they know from their own thinking that it is so, they 'gain ownership'.

Having results linked via the process of logical deduction helps demonstrate that mathematics is not a collection of isolated facts but a coherent whole.

Proof is an important part of what mathematics is. No one should be considered cultured or educated unless they have some understanding of what proof entails.

Quite apart from its role in mathematics, logical and deductive reasoning is valuable in a very wide range of professions. Its role in society as a whole in encouraging a healthy scepticism in young adults is also important. The need for clear reasoning arises when it is considered unwise to accept what is offered at face value. In an increasingly technological society where access to snippets of information is extremely easy, it is very important that tomorrow's citizens have a questioning disposition. It is partly the importance of the skills of logical reasoning that has led to the central position of mathematics in the curriculum over many centuries.

We hope it is becoming clear that, although we believe proof to be important in school mathematics, we do not regard teaching and learning about it as straightforward. The teacher will need both skill and patience in leading the pupils gradually from early and acceptable simple explanations of an informal nature towards more precise argument. The presentation of proofs at a level of abstraction beyond a pupil's current understanding will achieve little. Indeed it should be accepted that not all pupils will reach the level of being able to construct formal proofs.

3 Proof and geometry

One might wonder why a report on geometry should focus heavily on proof. As many comments from respondents confirmed, Euclidean geometry was traditionally the area of mathematics within which proof was first encountered in a serious way, although this has not been as true in recent years. It is true that proof is both possible and desirable in other parts of school mathematics; and indeed that some types of proof are not really geometric in nature. But there is a case that, for some pupils, geometry is a suitable area of school mathematics in which to begin handling proofs. The arguments in this direction were summed up neatly for us by one of our number, Tony Barnard, in the following ten points, (some of which can be found in JE McClure, 'Start Where They Are: Geometry as an Introduction to Proof', *American Mathematical Monthly*, January 2000).

Ten reasons why geometry is particularly suitable for developing skills in mathematical thinking:

i) ***Familiar objects***. Geometry enables pupils to engage in proof (and with a system of logically connected material) in a concrete setting where the objects of attention - angles, parallel lines, triangles, circles - are already familiar. In other areas where pupils meet proof, they often have to cope with additional difficulties such as the meaning of symbols, abstract statements and quantifiers.

ii) ***Description rather than definition***. The activity in geometry at school level takes place in a world where things are 'already there'. The properties and relationships have to do with pre-existing mental objects, rather than objects (such as a mathematical group) which owe their existence and properties to an abstract definition.

iii) ***Accessible statements.*** The statements made are readily intelligible. For example, there is an immediacy about the statement, "the angles of a triangle add up to 180°", that is less evident in the statement, "if $b^2<4ac$, then the equation $ax^2+bx+c=0$ has no real roots".

iv) ***Straightforward logic.*** The logical methods involved in basic plane geometry tend to be less subtle than those in other introductory parts of mathematics; for example, they involve fewer quantifiers and the 'indecomposable statements' are generally less complex.

v) ***An early start.*** In view of the points above, pupils are able to use and develop their skills in logical thinking as soon as they emerge, rather than wait till a later stage in their mathematical education. This can have advantages for their intellectual development generally (and disadvantages if delayed).

vi) ***Synthetic deduction.*** Whereas most deduction in school mathematics takes the form of a linear sequence with each conclusion following from the previous one (a single dominant cue prompting a closed procedure), basic plane geometry involves contemplating several statements at the same time, reorganising them and drawing conclusions from the collection as a whole.

vii) ***Route finding in solving problems.*** Geometry is extremely suitable for developing the skills of recognising and aiming for fruitful intermediate objectives, providing an ideal visual and verbal scenario for considering where you are, where you want to go, and how you might get there. In geometry it is relatively easy to set a problem for which (a) the solution is not immediately obvious or reachable by a memorised algorithm, but (b) a very little bit of playing about with the problem (for example, marking all the angles and sides that you can establish to be equal) will enable the pupil to find the solution.

viii) ***A taste of higher mathematics.*** It is possible to do serious mathematical learning in geometry without having a perfect understanding of what axiom systems are and what the rules are for working with them.

ix) ***Not having to take things on trust.*** It has been said that one of the unique features of mathematics is that it is the only subject where you don't have to take things on trust. Geometry is well suited for demonstrating that mathematics is not something handed to pupils by an authority (whether that is a person, a book or a computer), but is something they can find inside their own heads, and that when they have found it they will have access to a system of truths which is entirely different in nature from any other system. Not only is it important to be able to tell the difference between something which is proven and something which is not, but a mathematical education should, at the very least, include the realisation that there is this dimension to knowing.

x) ***Surprising effectiveness.*** In geometry, there is the situation where a small number of plausible assumptions lead to a large number of surprising and appealing results. Isaac Newton expressed this point well when he wrote: "*It is the glory of geometry that from so few principles, fetched from without, it is able to accomplish so much*." *Philosophiae Naturalis Principia Mathematica*, Praefat.

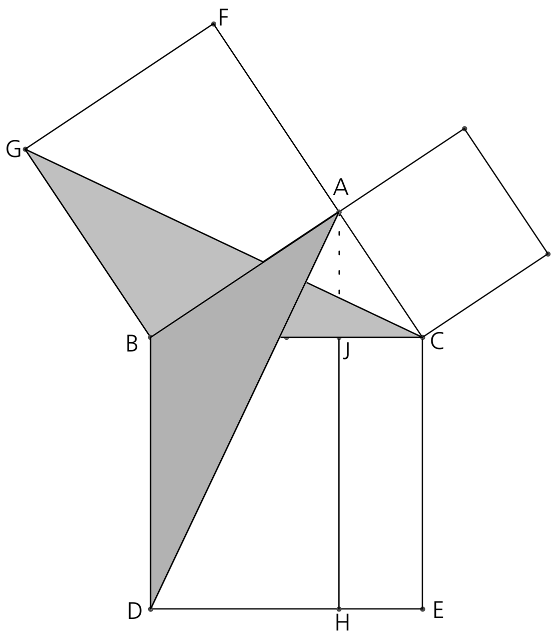
4 Examples of geometric proof

Now we illustrate some of these ideas through a number of examples. Further examples are given in Appendix 11.

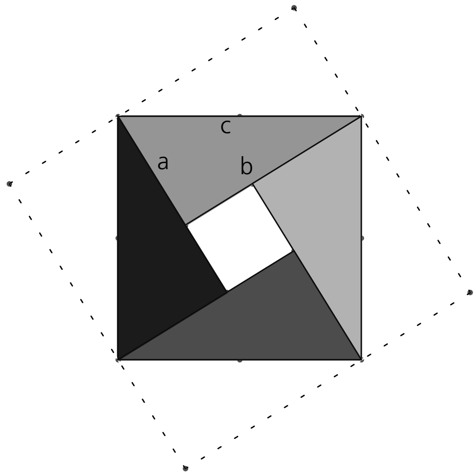
a) Pythagoras's theorem

There are many proofs, and an interesting task is to try and collect some. Here are four.

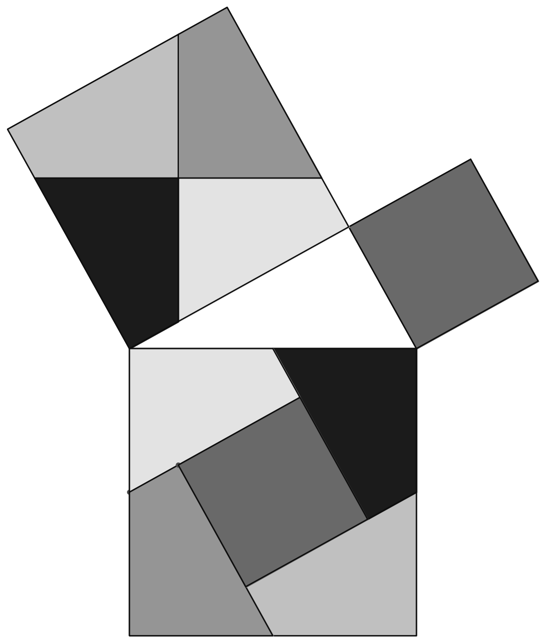
(i) The first is essentially quite visual in nature. The triangle *ABD* is rotated through 90˚ about *B*. Why is its image identical with triangle *GBC*? How does this help show that the area of the square ABGF is the same as that of the rectangle *BDHJ* where *J* is the point of intersection of *AH* and *BC*? How does that help prove Pythagoras?



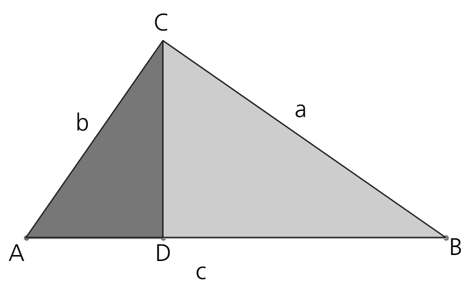
(ii) The second is a more algebraic proof. It involves noting that the small square together with four copies of the right angled triangle fill the next size square, thus $c^2 = (b\text{-}a)^2 + 4.(ab/2)$. When simplified, this gives $c^2 = b^2 + a^2$. Similarly the largest square, shown with dotted lines, is made up from the square of side *c* together with four of the right-angled triangles. This gives another means of derivation.



(iii) The third is more visual, showing dissections of two of the squares. The challenge here is to demonstrate that the non-square quadrilaterals are congruent, and so have the same area.



(iv) Finally, a proof of Pythagoras which uses similarity of triangles. The triangles *ABC*, *ACD* and *CBD* are all similar and so their areas are in proportion to the squares of their corresponding sides, i.e. $c^2 : b^2 : a^2$. Hence, for some $k \neq 0$, $k\,a^2 + k\,b^2 = k\,c^2$ and Pythagoras follows.



b) The regular solids

A very different style of proof, that of proof by exhaustion, is provided by Sir Christopher Zeeman. This is an example of a proof in a 3-D context which is accessible to students with a wide range of ability.

Definition
A regular solid is a convex solid which has all its faces equal to the same regular polygon, and the same number of faces at each vertex.

Theorem
There are exactly 5 regular solids.

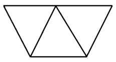
Proof
Given a regular solid, the ring of faces around a vertex contains at least 3 faces, and, if the ring is cut open along an edge and flattened out, it will occupy strictly less than 360 degrees. If the faces are equilateral triangles the ring can contain only 3, 4, or 5 triangles because 6 would occupy the full 360 degrees; there are 3 cases:

If the faces are squares there is only one case because 4 squares would occupy 360 degrees:

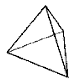
If the faces are pentagons there is similarly only one case:

There are no more cases because 3 hexagons (or higher polygons) would occupy 360 degrees (or more).

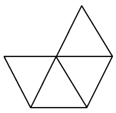
3 triangles



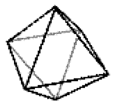
giving a tetrahedron



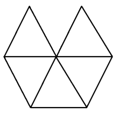
4 triangles



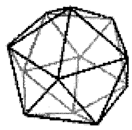
giving an octahedron



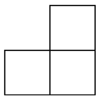
5 triangles



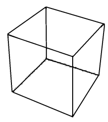
giving an icosahedron



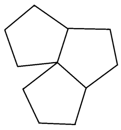
3 squares



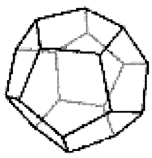
giving a cube



3 pentagons

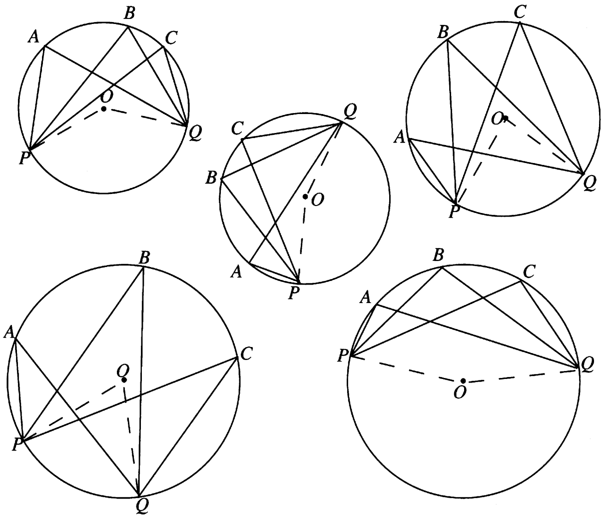


giving a dodecahedron



c) Circle theorems

The third set is taken from the Mathematical Association's book *Can you prove it* ? by Sue Waring and shows possible approaches to one of the circle theorems.



You may find it helpful to extend the arms of some angles when measuring them.

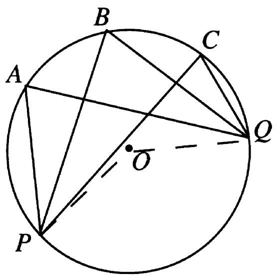
In each diagram above angles A, B, C, and O are standing on (subtended by) an arc PO of a circle centre O.

1. In each diagram measure angles, A,B and C and record the results in a table:

Diagram	A	B	C
1			
2			

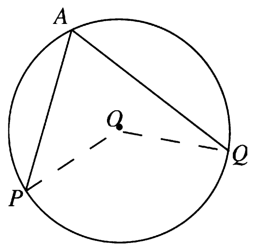
2. What seems to be true in each case?
3. In each diagram draw another angle standing on an arc PO and label it D. Predict the size of angle D and then measure it. If your predicted and measured values agree record them as column D in the table. If not, re-measure angle D.

Diagram 6



4. In diagram 6 measure angle A and predict the size of angles B and C. Check your predictions. On what basis did you make them?
5. In all diagrams measure the angle at the centre of the circle marked O, and record in your table, as column O.
6. What seems to be true in each case?

Diagram 7

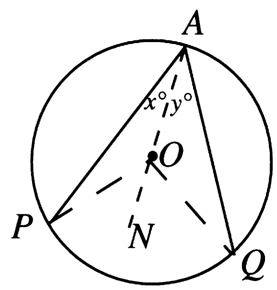


7. In diagram 7 measure angle A and then predict the size of angle O. Check your prediction by measuring. On what basis did you make your prediction?
8. There is a connection between correct answers to questions 2 and 6. What is it?

You now have two related conjectures arising out of results of measurements in seven circles.

To prove: the angle at the centre of a circle is equal to twice any angle at the circumference standing in the same arc.

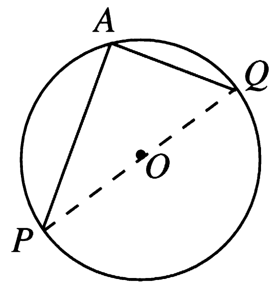
Proof



This diagram has the same form as diagram 7 above with the line AON through the centre of the circle, added. It divides angle PAQ into two parts of sizes x° and y°. Write down the size of angle PAQ.

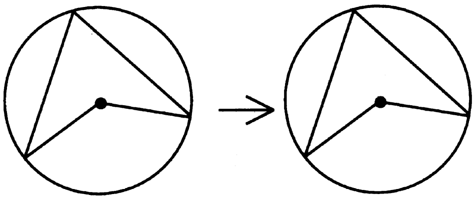
What can you say about OP, OA and OQ?…. Why?
What kinds of triangles are OAP and OAQ…. Why?
What are the sizes of angles OPA and OQA?….Why?
What are the sizes of angles AOP and AOQ?…. Why?
What are the sizes of angles PON and QON?….Why?
Write down the size of angle POQ.
What is the connection between angle POQ and angle PAQ?
Was angle A special in any way?
Would the above argument apply to any angle at the circumference?
What can you deduce about such angles?

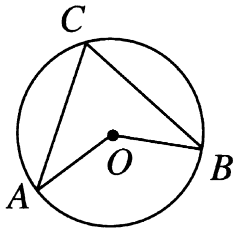
EXTENSION
In this diagram PQ is the diameter of the circle.
What is the size of angle POQ?…Why?
What can you deduce about angle PAQ….Why?
Write down a general conclusion.



Theorem: the angle at the centre of a circle is twice any angle at the circumference (standing on the same arc).

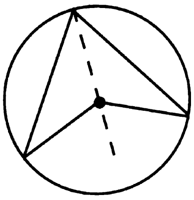
To Prove

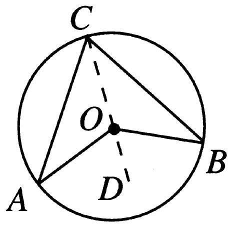




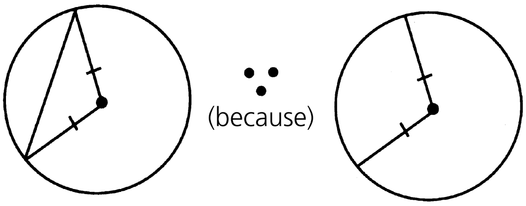
$A\hat{O}B = 2 \times A\hat{C}B$

Proof

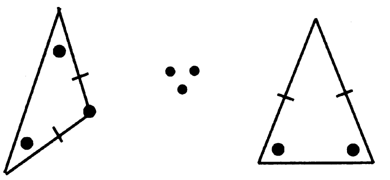




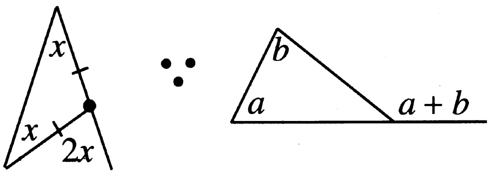
Join CO and produce to D



CO = AO (radii of circle)

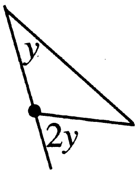


$O\hat{C}A = O\hat{A}C$
(base angles of isosceles triangle)



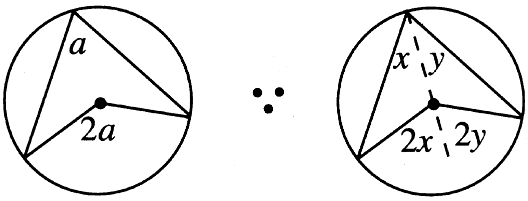
$A\hat{O}D = 2 \times A\hat{C}O$
(exterior angle of a triangle equals the sum of interior opposite angles)

Also



Similarly

$D\hat{O}B = 2 \times O\hat{C}B$



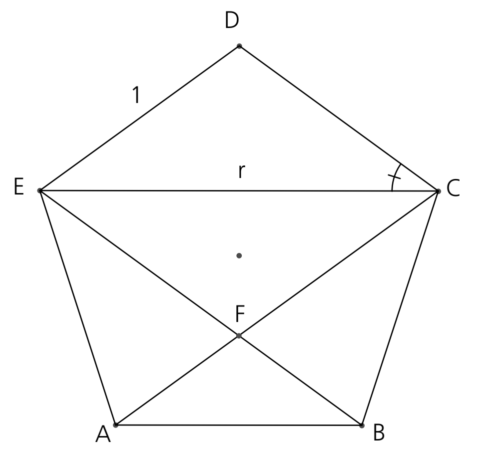
and $2x + 2y = 2(x + y)$

$A\hat{O}B = A\hat{O}D + D\hat{O}B$
$= 2 \times A\hat{C}O + 2 \times D\hat{C}B$
$= 2 \times (A\hat{C}O + D\hat{C}B)$
$= 2 \times A\hat{C}B$

d) Golden ratio and the construction of a pentagon / pentagram

Finally, we present a short outline by Adrian Oldknow which connects some geometric and algebraic ideas involved with the golden ratio. The golden ratio (1+√5)/2, or about 1.618, is also known as the 'divine proportion', and is often associated with beauty. Golden ratio is also intimately connected with the regular pentagons.

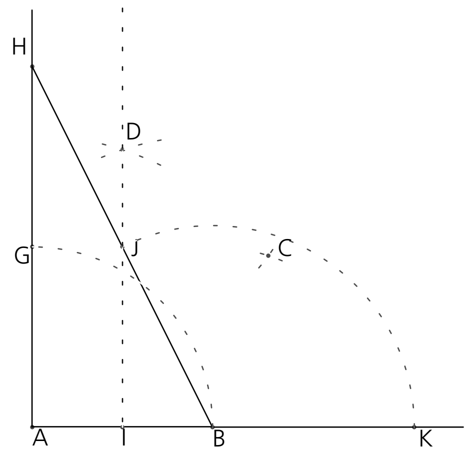
The diagram below shows a regular pentagon *ABCDE* together with three of its diagonals. The sides of the pentagon are taken as length 1, and the first objective is to find the length *r* of a diagonal (with *r* > 1). The internal angles of the pentagon are each 108˚ (why?), and hence the angle marked at *ECD* is 36˚ (why?). Can you now find the size of every angle in the figure below? There are many isosceles triangles, some congruent like *ECD* and *CAB*, some similar like *EFC* and *BFA*. There is also a rhombus, *CDEF*, and trapezia like CEAB. Can you mark any pairs of line segments which are parallel? Can you explain why they must be parallel?



Can you explain why *AFE* is an isosceles triangle?
Given that it is, then *EF* = 1 and *FB* = *r* - 1
Now consider the similar triangles *FCE* and *FAB*. *FCE* has sides 1,*r*,1 and *FAB* has sides *r*-1,1,*r*-1 . As the ratios of the sides must be the same we have: 1/*r* = (*r*-1)/1 .
Can you rearrange this to give a quadratic equation in *r* ?
Given that *r*>1 , can you show that *r* = (1 + √5)/2 ≈ 1.618 ?

Now the challenge is to work backwards to see if given a side *AB* of length 1 we can use this information to construct a 'perfect' pentagon. Now the ancient Greeks could use their geometry in much the same way as we use a calculator today. How do you think Pythagoras might have drawn a length of √5 if he already had a segment of length 1?
Can you suggest what sorts of right-angled triangles might have a hypotenuse of length √5?
How to you think he might have added two numbers geometrically, such as 1 and √5?
How could he have used a construction to do the same job as dividing by 2?

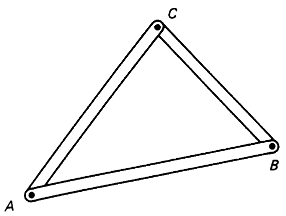
In the construction below we illustrate one way of putting this theory into practice.
AB is the side of length 1. We construct the perpendicular to *AB* at *A* so that we can make a right-angled triangle. We want to find a point *H* on this line so that *AH* = 2, so using compasses we can first find *G* so that *AG* = 1 by drawing the circle centre *A* through *B* to cut the perpendicular. Then we can use compasses again (how?) to find *H*. Now *BAH* is a right-angled triangle with sides 1 and 2, and hence a hypotenuse of √5. In order to divide by 2 we need to find mid-points. Construct *I* as the mid-point of *AB*, so that *IB* = 1/2. The perpendicular bisector of *AB* cuts *BH* in *J* where *BJ* = √5/2. Now we just need to 'add' *IB* and *BJ* together, so we need to 'swing' *BJ* round to line up with *IB*. Can you see how to do this? The segment *IK* now has the required length *r* . Using compasses with this as radius can you see how to construct the vertex *D* of the pentagon? How can you find the other two vertices *C* and *E*? So now you can complete the pentagon, and, by drawing all its diagonals, also create a pentagram. What shape appears in the middle of the pentagram? How are its sides related to the sides of the bigger pentagon?

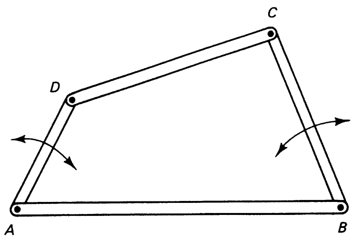


Appendix 10: Examples of applications of geometry

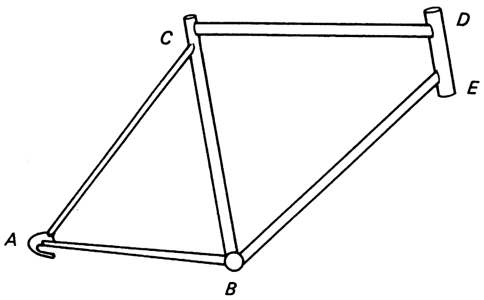
There are many examples to be found in familiar rigid structures and in mechanisms. Many of these have been collected in a form suitable for classroom use by Brian Bolt. The following extract on rigid structures is taken from Brian Bolt's submission to the working group.

1 Rigid structures



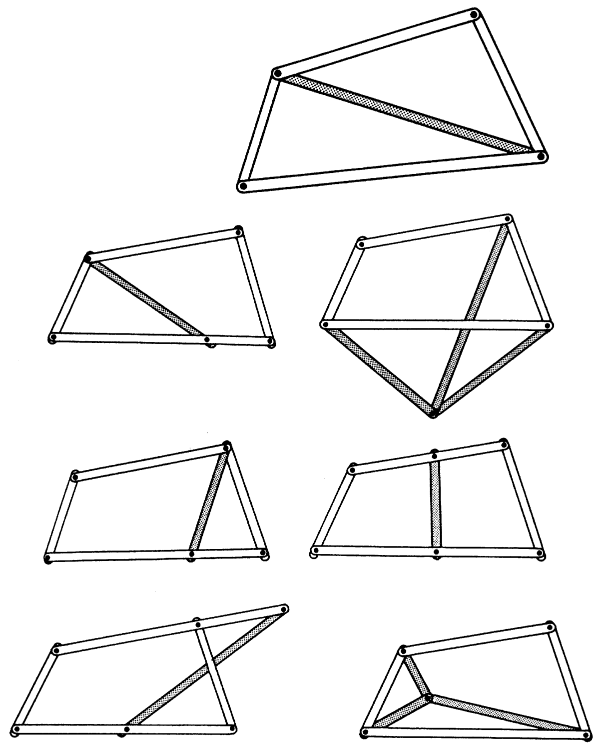


Make a triangle linkage from plastic geostrips or card strips using paper fasteners to join their ends. If you place the linkages on a table and fix AB then the point C or the triangle linkage is also fixed, but points C and D of the quadrilateral linkage are free to move. This illustrates the innate rigidity of the triangle linkage, but lack of rigidity of the quadrilateral linkage. The traditional cycle frame makes use of the triangles strength at the rear, but the main part of the frame BCDE, which supports the front forks, is dependent on the strength of the welds in the joints.

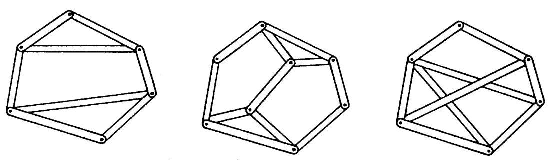


If you are observant you will not have to look far to see places where the rigidity of the triangle is used. Look at a folding chair, the fastening which hold a window open, the roof timbers in a house, an umbrella, a rotating clothes airier, the legs of an ironing board, the design of a traditional 5-bar gate, just to name a few.

To increase understanding of which two-dimensional frameworks are intrinsically rigid an investigation can be made, for example, of what struts can be added to a quadrilateral in order for it to maintain one shape. It is soon clear that one of the diagonals will suffice but how about all the other possibilities suggested by the following drawings. A traditional geometrical education gives little help in this and it is essential to make models to come to terms with the problem.



Having decided which of the above frameworks are rigid the investigation can be followed up by considering pentagonal or hexagonal frameworks. How about the following? Are any of them rigid? What is the minimum number of struts required to ensure that an n-gon framework is rigid?

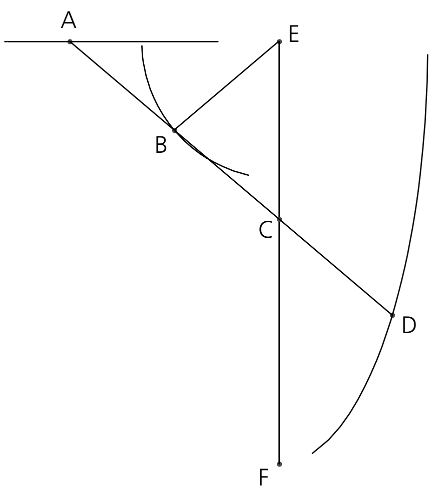


This concept of intrinsically rigid framework should be extended into three dimensions. There are so many structures in our modern society where they can be observed. Tower cranes can be seen like exotic creatures looking down on the buildings being built beneath them. The structures of their towers and their jibs are prime examples of rigid frameworks as are the electrical pylons which stride like giants across the countryside. Many bridge structures such as the famous Forth Bridge in Scotland or the Sydney Harbour Bridge in Australia are well known examples but so are the many railway bridges from the pioneering days when they were made of timber to the metal ones of today. The Eiffel Tower in Paris is such a structure, but on a small scale look at the scaffolding on a building site or visit a fun fair to see exciting examples.

2 Garage door

An example of modelling the mechanism for an 'up and over' garage door is given in [Oldknow and Taylor, 2000]. This provides a good example for the study of locus.

Figure 1

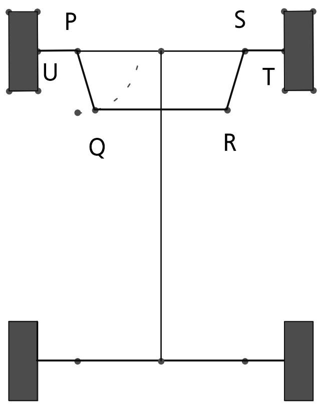


Here the door is represented by *AD*. The bar *EB* is free to rotate about the top of the door frame *E*, and a fixed pin at *B*. The pin at *C* is free to slide inside a groove along *EF*. A model can easily be made using drawing tools, cardboard strips or geometric software. Using dynamic geometry the loci of points such as *A*, *B* and *D* can be drawn. Given that $BA = BE = BC$ geometric reasoning can be used to establish the loci of *A* and *B*. At A-level, coordinate geometry can be used to find the coordinates (x,y) of *D* in terms of the angle $\theta = BEF$ and the lengths $a = AB$ and $b = CD$, and hence to deduce the equation of the locus of *D* as an arc of an ellipse.

3 A car steering mechanism

The following is adapted from *Teaching Mathematics with ICT* by Adrian Oldknow and Ron Taylor.
The system, known as Ackermann steering, is based on a trapezium. When the front wheels are pointing straight ahead, the quadrilateral *PQRS* forms a trapezium. The 'stub axle' *UP* makes a fixed angle with the 'track-rod end' *PQ*, and they both pivot about the fixed point *P* (the 'king-pin'). Similarly for *R,S* and *T*. Points *Q* and *R* are joined by a rod, called the tie bar, which can pivot loosely at *Q* and *R*. If *Q* is moved on a circular arc centre *P*, so *R* describes a circular arc centre *S*. For a given wheelbase *UT* and length between axles, the shape of the trapezium is defined by the two parameters $p = PQ$, the length of the track rod ends, and $q = QR$, the length of the tie bar. As *Q* slides on the arc centre *P*, the stub axles *PU* and *ST* turn through different angles. (They would be the same if *PQRS* was a parallelogram.)

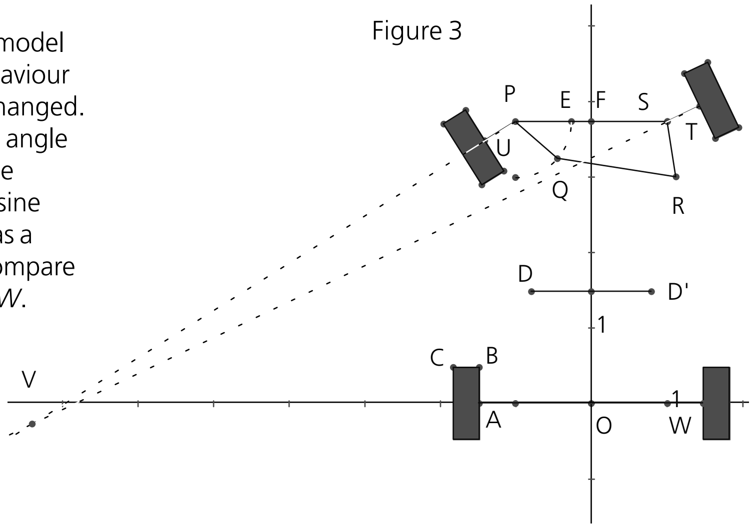
Figure 2



Now it is highly desirable that when taking a bend, the four circles to which the tyres are tangents should all have the same centre - otherwise the front tyres will soon lose their tread. The design problem is to choose *p* and *q* so that the point *V* of intersection of the stub axles produced lies as close to the line *AW* as possible for all positions of *Q*. Of course there also physical constraints on the maximum sizes of *p* and *q*.

The diagram above suggests making a dynamic model e.g. in *Cabri* from which you could study the behaviour of the locus of V as the parameters p and q are changed. You could also make an analytic model using the angle $QPS = \theta$ as independent variable, and splitting the quadrilateral $PQRS$ into two triangles. Using the sine and cosine rules you can find the angle $QRS = \phi$ as a function of θ (perhaps in a spreadsheet?), and compare it with the desired value ϕ' found when V is on AW.

Figure 3



4 A London day

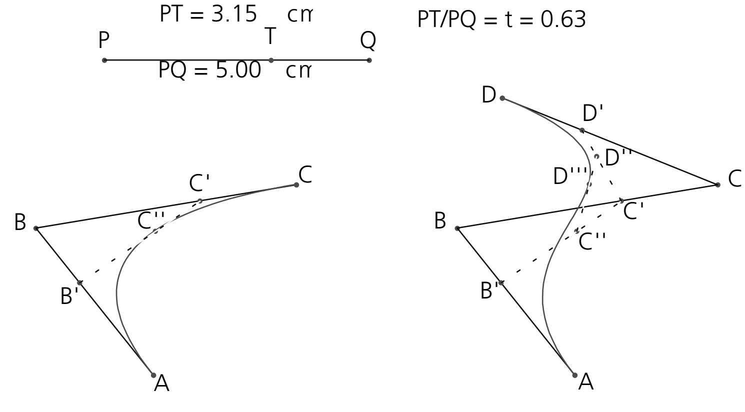
Now that digital cameras, or scanners for computers, are more or less commonplace a trip out can be used to capture a variety of geometric images which might act as stimuli for work in geometry in schools and colleges. Here are a set of photographs taken of the 'London Eye'. Can you suggest what sort of route the camera operator took while shooting these pictures? Why is it that a circular object appears elliptical when viewed from an angle. Can you find a way to use the ratio between the shortest diameter and the longest diameter of a picture of a circle to work out the angle the picture must have been taken from, measured from the axis of the circle?

5 CAD and Bézier curves

Design tasks used to be carried out using drawing boards. In order to draw smooth curves there was a device called a 'draughtman's spline' - where weights could be placed on the board, with grooves on their top, through which a flexible piece of steel, or laminated wood (called a 'spline') could be passed. As the weights were moved so the flexible curve could be controlled to take a desired shape. With the move to computer based design there have been several systems developed to produce a 'virtual' equivalent to the physical 'flexible curve'. One such fundamental form for representing flexible curves in Computer Aided Design is based on Bézier curves. Here $n+1$ points are defined on the screen which are used to define a unique n-th degree polynomial. If any point is moved, then the whole curve is changed. The points are called 'control points', and the effect is called 'global control'. Unlike some other systems for producing flexible curves, Bézier curves do not, in general, pass through the control points (other than the first and last). The curves are usually defined algebraically using binomial coefficients. However they can also be constructed by a set of dilations (also known as dilatations).

First we define a parameter between 0 and 1 by taking a point T which can slide on a segment PQ, and taking the ratio of PT to PQ as the parameter t (with $0 \le t \le 1$). Points A, B, C, D etc. are used to define the curve. The diagrams below show the special cases of a quadratic curve defined by three control points A, B, C and a cubic curve defined by adding a fourth control point D. The point B' on AB is the image of B when dilated with centre A and scale factor t. The points C' and D' are similarly defined. The point C'' on $B'C'$ is the image of C' when dilated with centre B' and scale factor. The locus of C'' as T slides on PQ is the desired quadratic. Points D'' and D''' are similarly defined, and the locus of D''' is the desired cubic.

The quadratic curve starts at A, tangent to AB and finishes at C tangent to BC. If a is the position vector of A, etc. then $b' = (1-t)a + tb$, and similarly for c' and d'. Hence $c'' = (1-t)b' + tc'$ from where you can show that the locus of C'' is quadratic in the parameter t, and deduce the claims about tangency. What can you say about the locus of D'' shown above? Now define D''' as the image of D'' when dilated with centre C'' and scale factor t. The position vector of D''' is the weighted average of the position vectors of C'' and D'' and so its locus is a cubic. Can you find its form in terms of t ? A, B, C and D need not be coplanar, and hence the locus of D''' can represent a 'twisted' curve in space, such as a section of a car's exhaust system. Cubic curves are frequently used as the basis for design systems since they are the simplest polynomials which can exhibit inflections.



Appendix 11: 3–dimensional geometry

contributed by Sir Christopher Zeeman

Given here are examples of topics in 3-dimensional geometry that have been successfully used in Royal Institution masterclasses. To illustrate how each topic might be taught I state and prove one or two theorems, followed by some exercises (with solutions given at the end). I have selected the theorems that are surprising, some classical and some modern, but all with short understandable proofs. I have chosen theorems that are essentially 3-dimensional, and will specifically enhance 3-dimensional thinking, which is an acquired skill useful throughout all branches of mathematics and science. The following five topics have been covered:

1. Perspective;
2. Regular solids;
3. Tetrahedra;
4. Spherical triangles;
5. Knots & links.

Topics 1,3 and 4 are found in this Appendix; an extract of topic 2 is contained within Appendix 9, and the whole of topics 2 and 5 are available *via* the Royal Society website at www.royalsoc.ac.uk

Assumptions:

(a) **Intersections**

In general:

- 2 planes meet in a line;
- a line meets a plane in a point;
- 3 planes meet in a point.

Exceptions occur when:

- the 2 planes are parallel;
- the line is parallel to, or contained in, the plane;
- the 3 planes are parallel, or the line of intersection of 2 of them is parallel to, or contained in, the third.

(b) **Two lines**

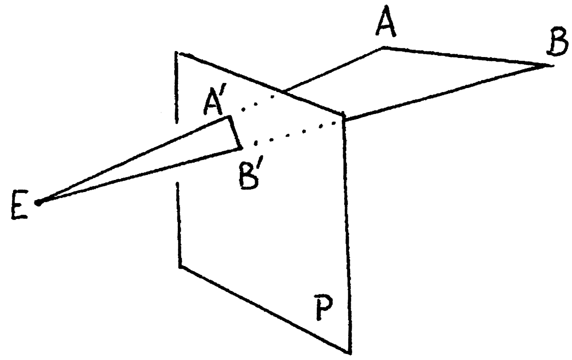
2 lines are contained in a plane if and only if they meet or are parallel. If 2 lines are not contained in a plane then they neither meet nor are parallel, and they are called *skew.*

Definitions of 'perpendicular'

- 2 meeting lines are perpendicular if they are at right angles.
- 2 skew lines are perpendicular if a line parallel to one and meeting the other is perpendicular to it.
- A line is perpendicular to a plane if it is perpendicular to 2 non-parallel lines in the plane, and consequently to every line in the plane.
- 2 planes are perpendicular if there is a line in one perpendicular to the other.

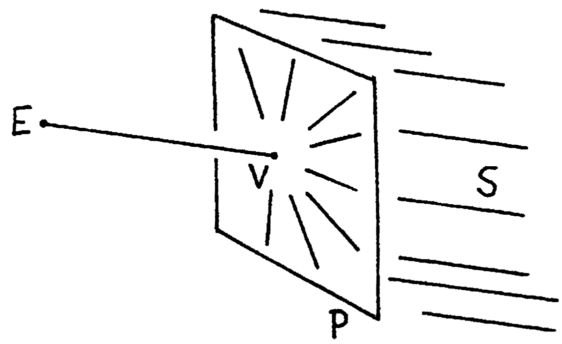
1 Perspective

Imagine painting a 3-dimensional scene on a pane of glass P placed in between the scene and the eye E.



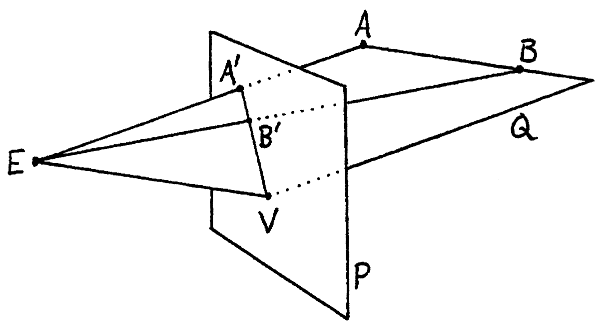
Definition 1: The image A′ of a point A is where the ray EA pierces P. If also the image of B is B′ then the image of the line AB is A′B′.

Definition 2: The vanishing point V of a set of S of parallel lines is where the parallel line through E pierces P.



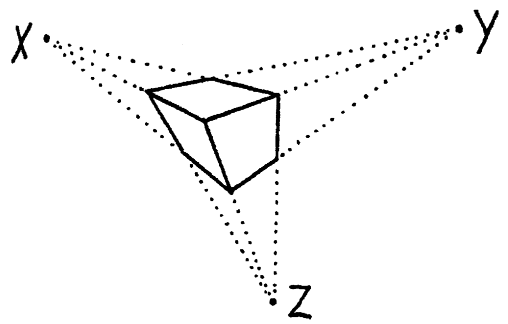
Theorem 1: All the images of S go through V.

Proof:



It suffices to prove that the image of one line, when extended, goes through V, because by the same proof they all will. Let AB be the line. Then EV is parallel to AB by definition 2, and therefore they both lie in a plane Q. The 3 points A', B' and V all lie in both the planes P and Q and hence on their line of intersection. Therefore extending A'B' along this line goes through V.

Drawing a cube



A cube has 3 sets of 4 parallel edges, and therefore a drawing of a cube needs 3 vanishing points. Choose an acute-angled triangle XYZ, and use the vertices as the vanishing points as shown. To then <u>see</u> the cube in perspective we must place the eye E in a position such that the lines EX, EY and EZ are parallel to the edges of the cube, which are perpendicular to each other. Therefore we define:

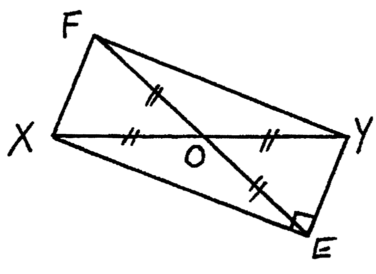
Definition 3: An observation point E is a point such that EX, EY and EZ are perpendicular to each other.

Theorem 2: There is exactly one observation point in front of P.

To prove the theorem we shall need the following lemma:

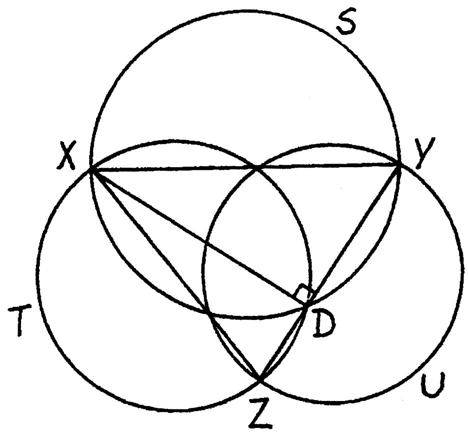
Lemma: If EX, EY are perpendicular then E lies on the sphere diameter XY.

Proof:



Complete the rectangle XEYF by drawing lines through X, Y parallel to EY, EX to meet in F. Let O be the intersection of the diagonals XY & EF. Then by symmetry OX=OE=OY=+OF. Therefore the circle centre O and radius OX is the circle diameter XY which goes through E. If we spin the circle about XY we obtain the sphere diameter XY.

Proof of Theorem 2:



Let E be an observation point. Let S, T, U be the spheres diameters XY, XZ, YZ respectively. Since EX, EY are perpendicular, E lies on S by the lemma, and similarly on T & U. Therefore we have to find the intersection of all 3 spheres. If C is the circle of intersection of S and T then we have to find the intersection of C with the third sphere U. Now X lies on C. Let D be the foot of the altitude from X to YZ. Then D lies on S because XDY is a right-angle. Similarly D lies on T and hence on C. Meanwhile D lies in between Y and Z because XYZ is an acute-angled triangle, and so D lies inside U. Meanwhile X lies outside U because YXZ is less than a right angle. Therefore C contains points both inside and outside U. Therefore C pierces U at 2 points. One of these points lies in front of P and the other is its mirror image behind P, because P is a plane of symmetry of all three spheres. Therefore there is exactly one observation point in front of P, as required.

Exercises

1. Prove the observation point lies in front of the orthocentre O of XYZ (ie that EO is perpendicular to P)
2. Draw on the board an equilateral triangle XYZ of side 1 metre. Use the vertices as the 3 vanishing points to draw some rectangle boxes in perspective. View from $1/\sqrt{6}$ metre in front of the orthocentre and confirm that the boxes all look 3-dimensional and rectangular.
3. Prove that if XYZ is obtuse-angled then there is no observation point.

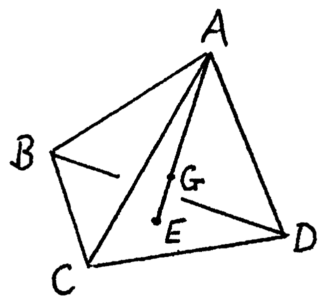
3 Tetrahedra

There are 4 theorems about 3 lines in a triangle meeting at a point: the 3 medians meet at the centroid, the 3 side-besectors meet at the circumcentre, the 3 angle-bisectors meet at the incentre and the 3 altitudes meet at the orthocentre. We shall show that three of these theorems can be generalised to a tetrahedron in 3-dimensions, but the fourth cannot.

Definition 1: A median of a tetrahedron is the line joining a vertex to the centroid of the opposite face.

Theorem 1: The 4 medians of a tetrahedron are concurrent at a point G.

Proof:



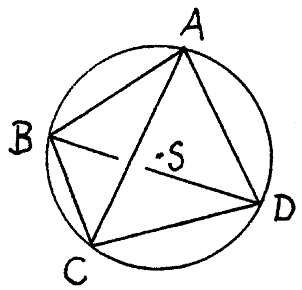
Let a, b, c, d be the vectors of the vertices A, B, C, D (with respect to some origin). Then the centroid E of BCD has the vector e=(b+c+d)/3
If G is the point with vector g=(a+b+c+d) /4 then g=a/4 + 3e/4. Therefore G lies on AE. Similarly for the other 3 medians.

Exercise 1: Show that G is the midpoint of each of the three lines joining the midpoints of opposite edges of the tetrahedron.

Definition 2: The bisector of a line AB is the plane perpendicular to, and through the midpoint of, AB; it is the set of points equidistant from A and B.

Theorem 2: The 6 edge-bisectors of a tetrahedron are concurrent at a point S, which is the centre of the circumsphere.

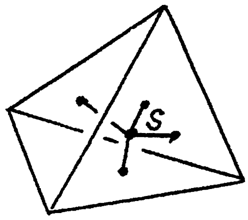
Proof:



Let the tetrahedron be ABCD. Let S be the meet of the bisectors of AB, BC and CD.
Then AS=BS since S lies on the bisector of AB;
BS=CS since S lies on the bisector of BC; and
CS=DS since S lies on the bisector of CD.

Therefore S is equidistant from all 4 vertices, and so the sphere centre S through one vertex is the circumsphere going through all 4, and S lies on every edge-bisector.

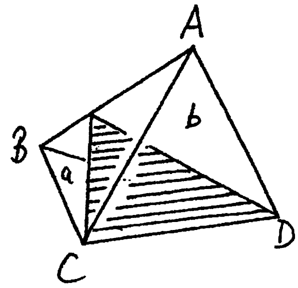
Exercise 2: Show that the 4 lines through the 4 circumcentres of the 4 faces, and perpendicular to those faces, are concurrent at S.



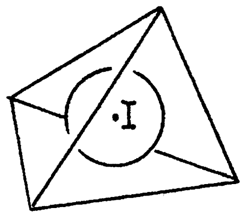
Definition 3: Let a, b, c, d, denote the faces of the tetrahedron opposite the vertices A, B, C, D.

Two faces a, b, meet in the edge CD: define the angle-bisector of ab to be the plane through that edge making equal angles with a and b; it is the set of points equidistant from a and b.

Theorem 3: The 6 angle-bisectors of a tetrahedron are concurrent at a point I,which is the centre of the insphere.

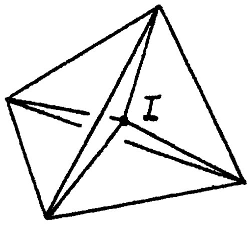


Proof: Let I be the meet of the angle-bisectors of ab, bc and cd. Then I is equidistant from a & b since it lies on the angle-bisector of ab, also from b and c since it lies on the angle-bisector of bc, and also from c and d since it lies on the angle-bisector of cd.



Therefore I is equidistant from all 4 faces, and so the sphere centre I touching one face is the insphere touching all 4, and I lies on every angle-bisector.

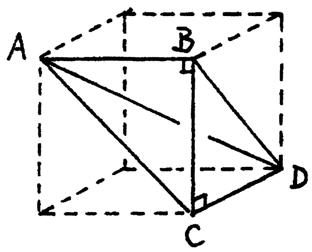
Exercise 3: Show that the 4 lines going through the 4 vertices, each equidistant from the 3 faces at that vertex, are concurrent on I.



Definition 4: An altitude of a tetrahedron is a line through a vertex perpendicular to the opposite face.

Theorem 4: In general the 4 altitudes of a tetrahedron are **not** concurrent.

Proof:



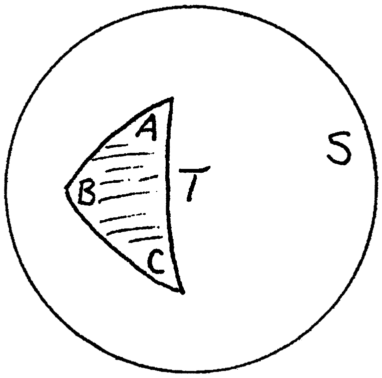
We construct a counterexample. Let ABCD be the tetrahedron inscribed in a cube as shown. Then the altitudes through A and D are AB and DC, which do not meet.

Exercise 4: Show that if each edge of a tetrahedron is perpendicular to the opposite edge then the foot of each altitude is the orthocentre of the opposite face, and the 4 altitudes are concurrent. Give two examples of such tetrahedra.

4 Spherical Triangles

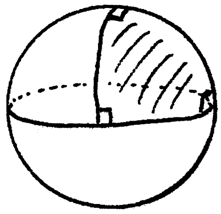
The theorem about the 3 angles of a triangle adding up to 180 degrees can be generalised to spherical triangles, and then used to give the sum of the 4 solid-angles of a tetrahedron.

Definition 1: A great circle on a sphere is the intersection of the sphere with a plane through its centre. A spherical triangle consists of 3 arcs of 3 great circles. Let A, B, C be the angles at the vertices (or more precisely between the tangents to the sides of each vertex). Let S = surface area of the sphere and T =surface area of the triangle.



Theorem 1: A+B+C = 180(1+4T/S)

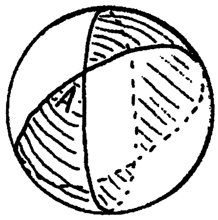
Example 1: The triangle shown has 3 right-angles and so A+B+C =270. Meanwhile T occupies a quarter of the northern hemisphere and so T/S =1/8

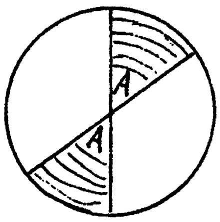


Example 2: If T becomes very small compared with S (like a triangle on the surface of the earth) then the sum of the angles tends to 180

To prove the theorem we need the following lemma:

Definition 2: Define the A-lune to be the area between the 2 great circles through A.

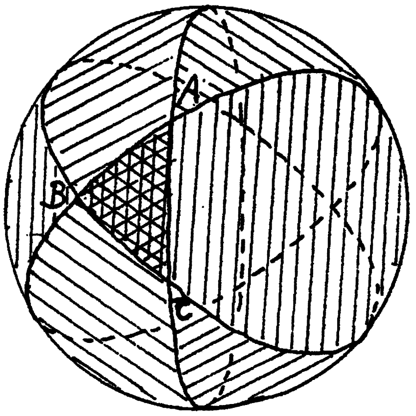




Lemma: A-lune /S = A/180

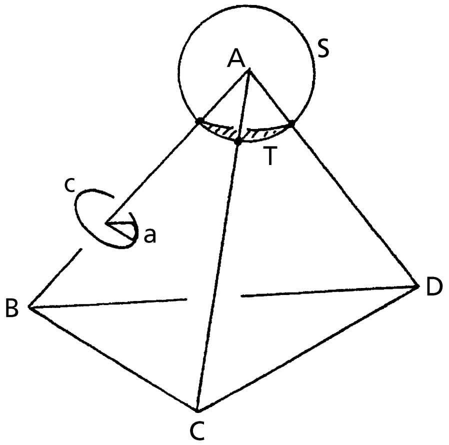
Proof: Looking down on S from above A
A-lune/S=2A/360=A/180

Proof of theorem 1:



The 3 lunes cover the whole sphere, but cover the triangle 3 times, which is 2 times too many, and the same with the antipodal triangle.
Therefore A-lune+B-lune + C-lune=S+4T
Therefore (A-lune +B-lune + C-lune)/S=1+4T/S
Therefore, by the lemma, (A+B+C)/180 = 1 + 4T/S
Multiplying by 180 gives the theorem.

Definition 3: In a tetrahedron ABCD define the solid-angle A to be T/S, where S is the area of a small sphere centre A, and T is the area of the triangle cut off by the tetrahedron.



Definition 4: Given an edge AB, define the dihedral-angle of AB to be a/c, where c is the length of the circumference of a small disc centred on and perpendicular to AB, and a is the length of the arc cut off by the tetrahedron. Notice that 1 unit of dihedral angle equals 360 degrees.

Theorem 2: In a tetrahedron, (the sum of the 4 solid-angles) =(the sum of the 6 dihedral –angles) – 1.

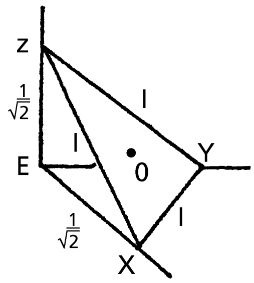
Exercises

1. Deduce Theorem 2 from Theorem 1.
2. Show that in a regular tetrahedron: dihedral-angles $= \cos^{-1}(1/3)$; solid-angles $= (3/2)\cos^{-1}(1/3) - 1/4$.
3. Calculate the dihedral and solid-angles of the tetrahedron used in the proof of Theorem 4 in Section 3.

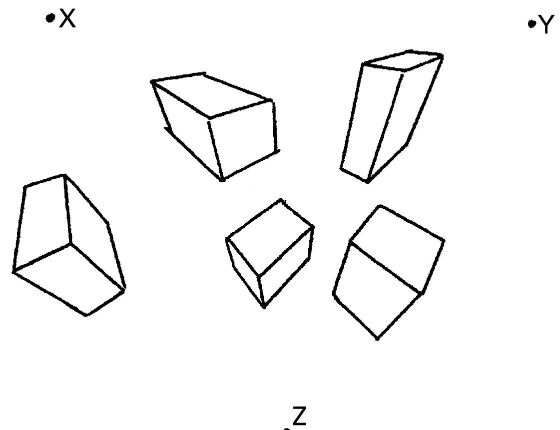
Solutions to the exercises

Section 1: Perspective

Exercise 1. Let E be the observation point, and O the orthocentre of XYZ. The plane containing the circle C is perpendicular to P and contains E and the altitude XD, and hence O and the line OE. Similarly the planes containing the other 2 circles of intersections of the 3 spheres are perpendicular to P and contain OE. Therefore OE is perpendicular to P, as required.



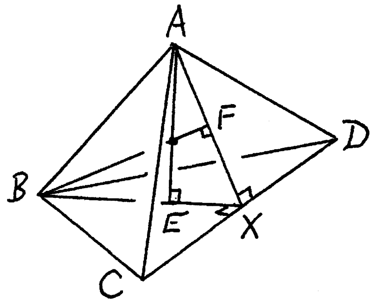
Exercise 2. With respect to axes EX, EY, EZ, the orthocentre O has coordinates (1/3)√2, (1/3)√2, (1/3)√2). Therefore EO=√(3/18)= 1/√6



Exercise 3. If X is obtuse then X lies inside the sphere U, along with D, and so C lies inside U. Therefore C does not meet U. Therefore the 3 spheres do not meet, and so there is no observation point. If Y or Z is obtuse then D lies outside U, along with X, and so C lies outside U. Again C does not meet U, and so there is no observation point.

Section 3: Tetrahedra

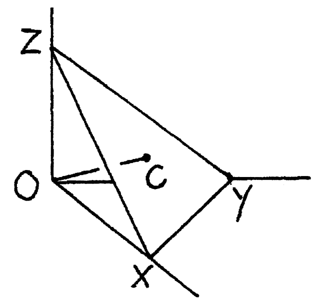
Exercise 1. Let X,Y be the midpoints of AB, CD. Then x=(a+b)/2, y=(c+d)/2 and so g=(a+b+c+d)/4 = (x+y)/2. Therefore G is the midpoint of XY.
Exercise 2. The line perpendicular to ABC through the circumcentre of ABC is the set of points equidistant from A, B, C, and therefore contains S. Similarly for the other 3 lines.
Exercise 3. The line through A equidistant from b, c, d, goes through I, and similarly for the other 3 lines.



Exercise 4. Let AE be an altitude of the tetrahedron, and suppose BE meets CD in X. Now AE is perpendicular to BCD, and so AE is perpendicular to CD. Meanwhile AB is perpendicular to CD by hypothesis. Therefore ABE is perpendicular to CD. Therefore BX is perpendicular to CD, and is hence an altitude of BCD. Therefore E lies on all the altitudes of BCD, and is hence the orthocentre of BCD.

Meanwhile AX is perpendicular to CD, and is hence an altitude of ACD, containing the orthocentre F of ACD. Therefore the altitude BF of the tetrahedron lies in the plane of ABE, and hence meets AE. Therefore all 4 altitudes of the tetrahedron meet pairwise, and are not coplanar, and so they must be concurrent.

Examples (i) The regular tetrahedron;
(ii) The tetrahedron OXYZ where X, Y, Z, are the unit points on the axes perpendicular OX, OY, OZ.

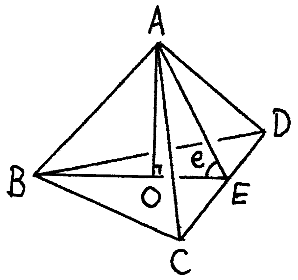


Let C be the centroid XYZ. Then the altitudes of the tetrahedron are OC, XO, YO, ZO, which are concurrent to O.

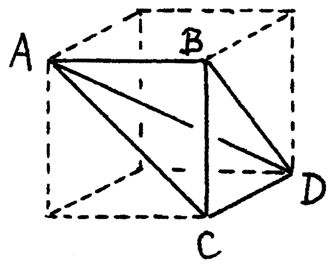
Section 4: Spherical Triangles

Exercise 1. Let d(AB)=dihedral-angle of AB, and s(A)=solid-angle of A. Then:
d(AB)+d(AC)+d(AD)=(1+4s(A))/2, by Theorem 1 (since 180 degrees equals half a dihedral unit). Summing over the 4 vertices repeats each dihedral-angle twice:
2(sum of the 6 dihedral-angles) = 2 + 2(sum of the 4 solid-angles).
Dividing by 2 gives Theorem 2.

Exercise 2. Let e be the dihedral-angle of a regular tetrahedron ABCD. Let E be the midpoint of CD, and O the centroid of BCD.



Then 3(OE)=BE=AE. Therefore cos e=1/3. Therefore e=$\cos^{-1}$(1/3). The solid angle = (6e-1)/4 = (3/2)$\cos^{-1}$ (1/3)-1/4



Exercise 3. Dihedral-angles AB, CD = 1/8
AD = 1/6
AC, BC, BD = 1/4
Solid angles A, D = 1/48
B, C = 1/16
Check: 2/48 + 2/16 = 2/8 + 1/6 + 3/4 - 1

Appendix 12: Framework for developing schemes of work for the curriculum

This appendix consists of an extract from a set of tables which is not a scheme of work but a framework. It was prepared by Richard Bridges, Margaret Brown, Sandy Cowling, Caroline Dawes, Jane Imrie, Mary Ledwick and Sue Pope. The complete set of tables can be found on the Royal Society website at www.royalsoc.ac.uk

It is a working document created by the teachers on our working group to show that the National Curriculum can provide the basis for a challenging and interesting geometry curriculum for all pupils.

The tables are organised according to the National Curriculum attainment levels (Key Stage 3) and GCSE grades (Key Stage 4). They highlight teaching opportunities including deduction and proof, and contexts and applications. They amply illustrate the level of detail which is required to plan a rewarding curriculum for pupils at all levels of attainment and can be used in creating schemes of work. They also remind us that the National Curriculum (a) does not tell teachers how to teach, or how to organise teaching, and (b) specifies a minimum curriculum to which confident teachers might well want to add further material, such as that on networks illustrated here.

The geometry curriculum for ages 11-14

CURRICULUM CONTENT							TOPIC	TEACHING OPPORTUNITIES			
Level 3	**Level 4**	**Level 5**	**Level 6**	**Level 7**	**Level 8**	**Exceptional Performance**		**Investigation and illustration**	**Deduction and proof**	**Use of ICT**	**Context and application**
Classify 2D shapes using angle and side properties.	Recognise congruence of shapes in different orientations.	Know and use the angle sum of a triangle.	Know and use angle, side, diagonal and symmetry properties of quadrilaterals. Know and use angle and symmetry properties of polygons.				**Euclidean plane geometry**	Angle sum of a triangle by practical demonstration. Angle sum of polygons by practical demonstration. Exploring properties of quadrilaterals.	Angle sum of a triangle using parallel lines. Angle sum of polygons using triangles.	Use of dynamic geometry to investigate and illustrate. Use of Logo to investigate polygons and stars.	History of Euclidean geometry. Tessellations. Fabric and graphic design.
	Distinguish between acute, right, obtuse and reflex angles.	Know and use the sum of angles at a point, and the sum of angles at a point on a line.	Know and use angle properties of intersecting and parallel lines.	Know and use the properties of tangents to circles.				Angle facts relating to intersecting and parallel lines using practical demonstration. The properties of tangents to circles. Tessellations.	Angle facts relating to intersecting and parallel lines.	Use of dynamic geometry to investigate and illustrate	
			Understand Pythagoras's theorem in the context of squares on sides of triangles.	Use Pythagoras's theorem to solve problems in 2D.		Use Pythagoras's theorem 3D.		Pythagoras's theorem Pythagorean triples.	Understand various proofs of Pythagoras's theorem.	Use of a spreadsheet to assist with investigation.	History of the relationship between the sides of right angled triangles (Babylonians, Chinese and Greeks). Spider and fly problems.

CURRICULUM CONTENT							TOPIC	TEACHING OPPORTUNITIES			
Level 3	**Level 4**	**Level 5**	**Level 6**	**Level 7**	**Level 8**	**Exceptional Performance**		**Investigation and illustration**	**Deduction and proof**	**Use of ICT**	**Context and application**
			Understand the concept of similarity and be able to identify similar shapes.	Appreciate the constant ratios of sides in similar right angled triangles. Use knowledge of similarity to solve problems.	Use trigonometry to solve 2D problems involving right angled triangles. Solve problems involving bearings and angles of elevation and depression.	Use right angled triangle trigonometry to solve problems in 3D.		Similar triangles leading to trigonometrical ratios.		Use of dynamic geometry to investigate and illustrate. Use of dynamic geometry to investigate and illustrate.	Bearings. Using angles of elevation & depression to determine heights & distances. Scale diagrams, maps & models. Ramps & slopes.
Use coordinates in the first quadrant.	Use coordinates in all four quadrants.			Calculate the length of a line segment given the coordinates of the end points. Understand the concept of gradient and use triangles to calculate gradient.	Understand and use 3D coordinates.		**Coordinate geometry**	Investigation of the distance between two points on a coordinate grid. Investigation of gradients.	Derivation of a general result for the distance between two points on a coordinate grid.	Use of dynamic geometry and graph plotters, including graphic calculators, to investigate and illustrate.	Locating positions on a map or grid. Air traffic control. Computer images in medicine & engineering. Link with algebra and $y = mx + c$. Link with distance time and velocity time graphs. Gradients of roads and slopes.

The geometry curriculum for ages 14-16

CURRICULUM CONTENT				TOPIC	TEACHING OPPORTUNITIES			
Grades G - F	**Grades E - D**	**Grades C - B**	**Grades A - A***		**Investigation and illustration**	**Deduction and proof**	**Use of ICT**	**Context and application**
Use coordinates in the first quadrant.	Use coordinates in all four quadrants. Find the coordinates of the midpoint of the line segment AB, given points A and B.	Calculate the length of a line segment in 2-D given the coordinates of the end points. Understand the concept of gradient and use triangles to calculate gradient.		**Coordinate geometry (continued)**	Investigation of the distance between two points on a coordinate grid. Investigation of gradients.	Derivation of a general result for the distance between two points on a coordinate grid.	Use of a graph plotting program. Use of dynamic geometry and graph plotters, including graphic investigate and illustrate.	Locating positions on a map or grid. Link with algebra and y = mx +c Gradients of roads and slopes. Locating positions on a map or grid. Air traffic control. Computer images in medicine and engineering.
Use distances to locate objects in 3D space.		Understand and use 3D coordinates.	Calculate the length of a line segment in 3D given the coordinates of the end points. Understand simple properties of points, lines and planes in 3D space. E.g. y = 4 is a plane, two planes generally intersect in a line.				CAD	
Construct nets of cuboids. Recognise nets of prisms, pyramids cylinders and cones.	Construct nets of pyramids, prisms, cones and cylinders from given information.			**3D geometry**	Use of equipment to make and handle models.		CAD	Design of packaging. Crystal structures. Cross curricular links with design technology.

CURRICULUM CONTENT				TOPIC	TEACHING OPPORTUNITIES			
Grades G - F	**Grades E - D**	**Grades C - B**	**Grades A - A***		**Investigation and illustration**	**Deduction and proof**	**Use of ICT**	**Context and application**
	Understand and use 2D representations of 3D objects including isometric drawings of shapes made from cuboids, simple sections, plans and elevations.	Explore polyhedra whose faces are regular polygons. Euler's rule F + V = E + 2	More difficult sections of solid shapes.		Investigation. Euler's rule. Existence of only 5 Platonic solids linked to regular tessellations.	Existence of only 5 Platonic solids.		Polyhedral forms in the natural world – pollen grains, viruses etc.
Carry out reflections of simple shapes in given mirror lines on Cartesian axes. Identify all lines of symmetry for 2D shapes.	Reflect and describe reflections of simple shapes in a range of mirror lines on Cartesian axes and use computer packages to reflect shapes.			**Symmetry, transformations and vectors**	Investigate two reflections being equivalent to a rotation.	Use rotations to prove Pythagoras's theorem.		Kaleidoscopes. Symmetry in art. Islamic design & architecture.
Understand and use vertical and horizontal displacement for location and movement.	Understand and use vector notation for translation and use computer packages to translate shapes. Describe combinations of translations as a single vector.	Understand and use vectors in the context of translation including inverse translation, repeated translations and combinations of translations.	Calculate and represent graphically the sum and difference of two vectors and a scalar multiple of a vector. Calculate the resultant of two vectors. Understand and use the commutative and associative properties of vector addition. Solve problems in 2D using vector methods.		Investigate repeated translations and associated vector arithmetic.			Transformations of graphs of algebraic and trigonometrical functions. Translations and tessellations.

CURRICULUM CONTENT				TOPIC	TEACHING OPPORTUNITIES			
Grades G - F	**Grades E - D**	**Grades C - B**	**Grades A - A***		**Investigation and illustration**	**Deduction and proof**	**Use of ICT**	**Context and application**
Enlarge shapes using any positive whole number scale factor.	Enlarge shapes using a centre of enlargement and any positive whole number scale factor.	Enlarge shapes using a centre of enlargement and any scale factor and describe enlargements.	Enlarge 3D shapes.	**Symmetry, transformations and vectors (continued)**	Investigate enlargement of shapes.			Combinations of rotations and enlargements to generate spirals.
	Solve simple problems involving enlargement of 3-D shapes.							Photographs.
								Desktop publishing.
Identify order of rotational symmetry.	Rotate a shape about the origin through multiples of 90°.	Rotate shapes using any centre of rotation and any specified angle and determine the centre and angle of rotation.						
Use practical equipment to investigate simple tiling patterns.		Demonstrate that any triangle will tessellate.			Investigate polygons and combinations of polygons that will and will not tessellate.			Tessellations.
		Determine which regular polygons will tessellate either singly (regular) or in combination with others (semi-regular).						
		Devise instructions for a computer to generate and transform shapes.					Use of Logo and dynamic geometry packages.	Pattern design.
		Apply simple combinations of transformations and describe the results using a single transformation.	Apply more complex combinations of transformations and describe the results using a single transformation.		Investigate combinations of transformations.			
		Distinguish properties that are preserved under particular transformations.						
			Stretch shapes using an invariant horizontal or vertical line and a scale factor.					

Appendix 13: Integrated approaches to geometry teaching

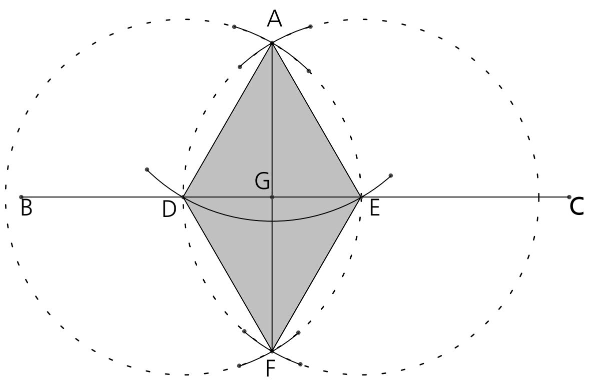
1 Introduction

We have already noted that the National Curriculum does not say how aspects of mathematics should be taught, nor, for example, do A/AS-level syllabuses. We are concerned that aspects of the geometry curriculum should not be taught in isolation. In this appendix we give examples of (a) how particular aspects of the geometry curriculum could be integrated within a particular theme, (b) where particular aspects of geometry could be linked with other areas of mathematics such as algebra and handling data and (c) where aspects of geometry could be linked with other subjects such as science, history and art.

2 Integration of aspects of geometry within a theme

A key to effective teaching of geometry is to combine experiential work with more formal argument in solving problems. As an example, consider straight edge and compass constructions, which are included in the Key Stage 3 programme of study. An approach sometimes encountered is to teach these constructions as a series of specific techniques, perhaps with some applications, such as finding the incentre or circumcentre of a triangle. A more fruitful approach could proceed as follows.

Certain questions are raised, such as '*What is the locus of a point which moves so that it is (a) an equal distance from two fixed points or (b) an equal distance from two fixed lines?*' These and similar questions are explored practically, for example, by getting pupils to stand in different places or using counters to represent points. In the course of this exploration, other questions will arise, for example, '*What is meant by the distance of a point from a line, and how is it found?*' (Dropping a perpendicular from a point to a line is another standard construction.)



Pupils become familiar with compasses as an instrument for constructing the locus of points which are a fixed distance from a fixed point. They recognise, for example when constructing a triangle given SSS, that the point of intersection of two circles is at specified distances from two fixed points.

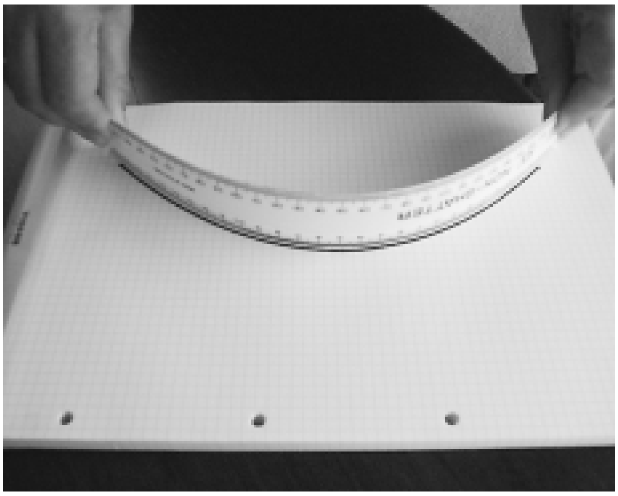
The practical exploration of loci described above motivates and provides a purpose for considering how to construct bisectors of lines and angles. The pupils' geometric awareness prompts consideration of how compasses can be used in construction.

The geometric aspects which are key to these and other basic constructions are those concerning the properties of the diagonals of a rhombus, namely that they bisect each other at right angles and also bisect the angles of the rhombus. If pupils have prior knowledge of these properties then, handled in an appropriate way, they can be used to approach the constructions as problem solving exercises, rather than a series of techniques to be learned.

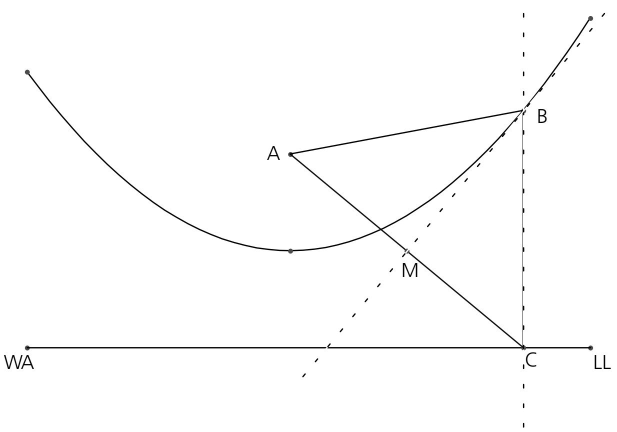
The benefits of this approach are that it develops problem solving skills, application of known results and reasons as to why particular techniques work. It also helps to integrate aspects of geometrical work (loci, properties of shapes, construction techniques) into a more powerful body of mathematics.

3 Integration of aspects of geometry with other areas of mathematics such as algebra and handling data

At Key Stage 4 pupils encounter quadratic functions in Ma2 *Number and algebra*. They also develop further ideas of construction, locus and coordinates in Ma3 *Shape, space and measures*. In Ma4 *Handling data* they collect data and represent it graphically. There are a number of familiar physical objects which appear to exhibit quadratic shape (i.e. that of a parabola) or its 3-D equivalent. These include bridges, bent rulers and satellite receiver dishes. Using modern technology images can be easily captured from the Internet, from photographs, or on digital cameras. Even without such technology curves can be traced and coordinates read off from a suitable grid.



The locus examples above can be extended to considering the path of an object which moves such that its distance from a fixed point is the same as that from a fixed line. For example in the classroom or playground a straight wall can be chosen as the fixed line. Then one pupil, called *A*, can stand about half way along the wall and say 2 metres away from it. Now the class can try to direct another pupil, called *B*, to move so that his/her distance from *A* is the same as his/her distance (measured perpendicularly) from the wall. In order to turn this into a construction suitable for use on paper, or with computer software, suppose *C* is any point on the wall. We need to construct the point *B* such that *AB* = *BC*. But we also know that *B* must lie on the perpendicular to the wall at *C*. Now if *ABC* is to be an isosceles triangle then we know that the perpendicular bisector of its base *AC* must pass through the vertex at *A*. Hence we just need to find the intersection of the perpendicular bisector of *AC* with the perpendicular to the wall at *C*. Now by taking several positions of *C* (or just by `dragging' *C* using dynamic geometry software) we can find the locus of *B*.



Using sophisticated language we can see that in the above construction the line (*directrix*) has been replaced by a line segment. This is the *domain* of the *independent variable C*. The distance of the point *A* (*focus*) from the wall is a *parameter* of the problem. The point *B* has been defined by constructions using the points *A*, *C* and the wall so that it is a *dependent variable*, and by letting *C* track through its domain we can find its *locus* with respect to *C*. So through geometry we can create images of functional relationships.

If we now take the wall as the *x*-axis, and its perpendicular through *A* (0, 2*a*) as the *y*-axis we can give the point *B* the coordinates (*x,y*) and use Pythagoras's theorem to find a relationship between them. The origin *O* (0,0) is the point on the wall nearest *A*. Let *D* be the point on *BC* so that *AD* is parallel to the wall. Then in the right angled triangled triangle *ADB* we have $AD = x$, $DB = y - 2a$ and $AB = BC = y$. Hence we have $y^2 = x^2 + (y - 2a)^2$, which simplifies to: $y = x^2/4a + a$, showing it is a quadratic function. We can also interpret the geometrical effects on the graph of $y = x^2$ of multiplying by a factor (1/4*a*) and adding the constant *a*.

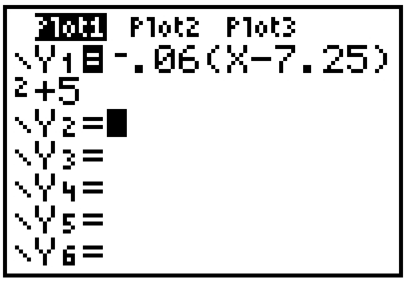
It also appears from the diagram that the perpendicular bisector of *AC* is a tangent to the parabola at *B*. Assuming this to be the case it is straightforward to derive the reflecting property of parabolas used in optical telescopes, and parabolic satellite dishes. Note: the lack of feasibility of a proof at this stage need not deter us from engaging with the activity. However, it is important not to gloss over such gaps in the logic but to emphasise them as unfinished business which will need to be resolved if the theory is to be watertight. At A-level the result can be proved using calculus to find the equation of the tangent to the parabola at *B*.

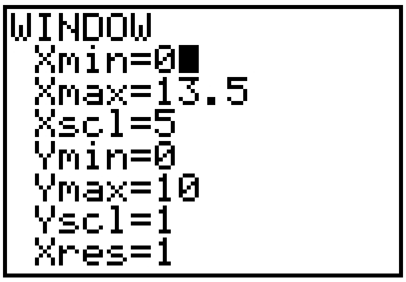
Another example uses an image as a source of data. Below there is a photograph of Sydney Harbour bridge. Tracing paper could be used to run over any of the curved sections from which coordinates could be read manually. Another way is to scan the photograph, or use a digital camera, to capture the image in picture editor software. As the cursor moves over the image a read-out of pixel coordinates is obtained automatically in the bottom right of the display.

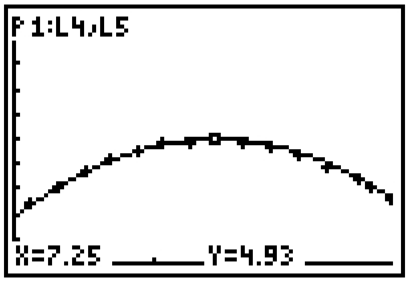
The table below shows the coordinates for a selection of points on the front lower curved arch.

x	58	152	252	348	440	533	627	725	817	918	1020	1122	1236	1285	1349
y	484	411	352	304	268	244	232	228	236	255	286	330	386	420	461

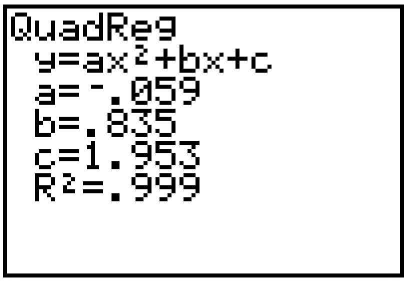
Of course the data are from an arbitrary origin and measured in pretty ghastly units. Data have been sampled for 15 of the 32 points where the vertical struts meet the lower front curved girder - which looks fairly like a parabola. The data can be transformed and displayed as a scattergram. A quadratic model can then be fitted by eye. For example, using a graphical calculator the data can be entered into lists L1 and L2. To transform the coordinates relative to an origin at the bottom left corner, which has pixel coordinates (0,721), 721-L2 can be stored in L3. Coordinates can be rescaled into units of, say, 100 pixels by storing L1/100 in L4 and L3/100 in L5. The scatterplot of L4 against L5 can then be drawn and quadratic functions superimposed by eye.

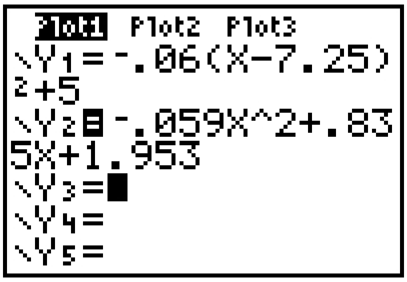


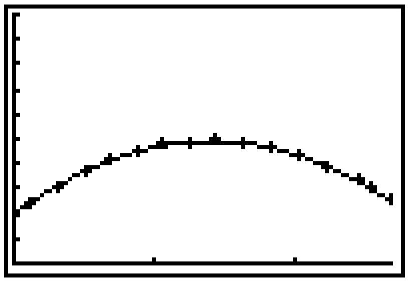




Many data handling packages, and graphical calculators, also provide the means of fitting models automatically. For example, quadratic regression gives a very good fit!







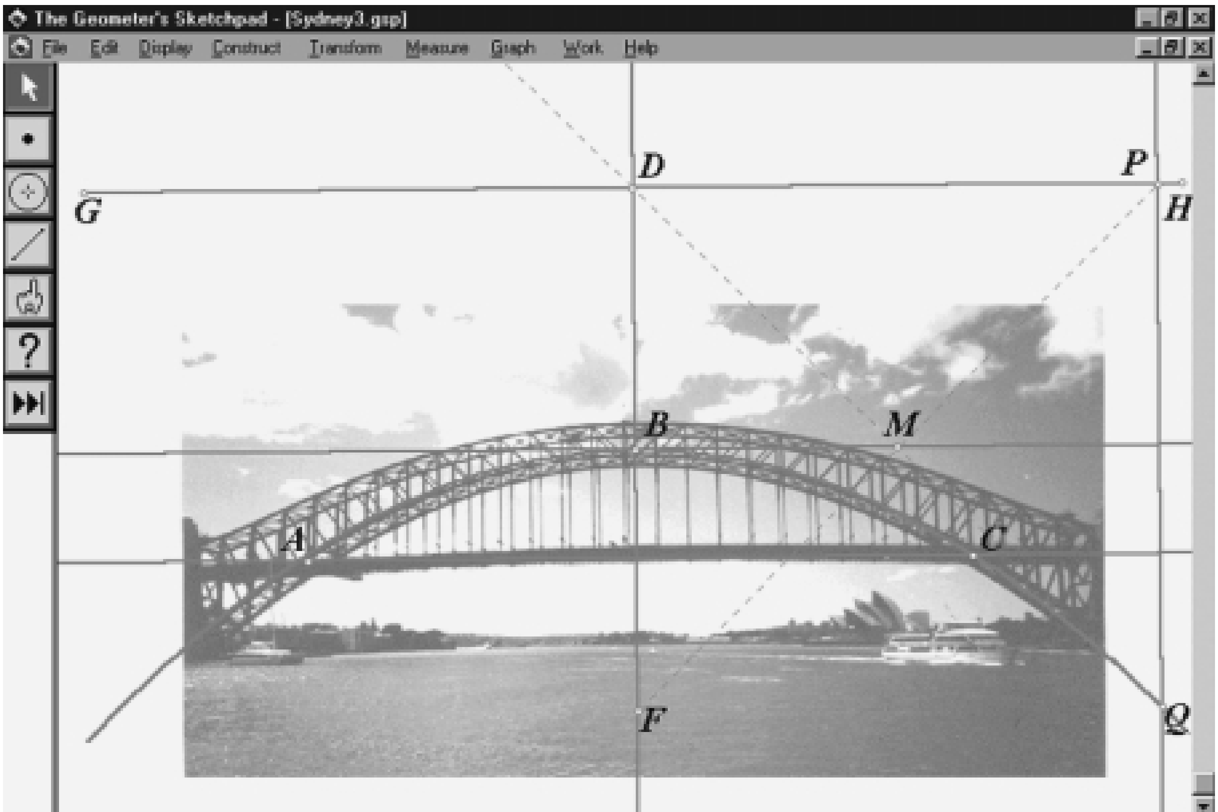
But it is also a nice exercise in algebra to equate the polynomial in the form: $ax^2 + bx + c$ with that in the form: $p(x - q)^2 + r$

This latter form is more convenient for modelling, and gives a live example of the general idea of transformations of functions f(x) in the form: $pf(qx + r) + s$. (Of course the transformation from $ax^2 + bx + c$ to $p(x - q)^2 + r$ is also the fundamental step in deriving the formula for the solution of a quadratic equation.)

Returning to the geometry, we note that here it is essential to get the aspect ratio right if the graph is in any way to match the photo! A nice exercise is to try to estimate the size of the new units in metres - zooming in on the original photo reveals a party of walkers nearing the flag at the top of the highest girder! Alternatively some research can be conducted, perhaps using the Internet, to find the span of the Sydney Harbour bridge. If we defined a measure called the 'bulge ratio', say, as b = (max(L5) - min(L5))/(max(L4) - min(L4)) - do you think that b is about the same (0.2) for all bridges of this type?

Try it on photographs of other bridges such as that across the Tyne in Newcastle. What about the cables on a suspension bridge - do they look like parabolas? Do they have a constant 'sag ratio'?

We can relate the image directly to geometric constructions, to test how well a parabola, or other curve, fits the bridge's shape. Here the image is pasted into a dynamic geometry package. Key points A, B and C are identified on the bridge. A and C are where the road meets the arch under question, and B is the highest point of the arch. The perpendicular through B to AC is constructed and F is taken as any point on it, (as in the next figure). D is the reflection of F in the line through B parallel to AC, and GH is part of the line parallel to AC through D. P is any point of GH. The perpendicular bisector of PF meets the perpendicular to GH at P in the point Q. The locus of Q as P varies is a parabola. F is the unique `control point' and as this is slid up and down so the shape of the parabola alters dynamically. There is only one position of F for which the parabola passes through A and C



An alternative geometric model for the curved arch could just be a circular arc through A, B and C - how would you find its centre? Can you detect any visible difference between the parabolic and circular models? Some mathematical analysis, as well as further images, can be found at http://www.brantacan.co.uk/ and many other images at: http://architecture.about.com/arts/architecture/

4 Integrating aspects of geometry with other subjects such as science, history and art

The 1996 OECD publication, *Changing the subject: innovations in science, mathematics and technology education*, contains accounts of a number of innovative projects in mathematics and science. An American project sought to integrate ideas in mathematics and science with the real world. Their basic tenet was that if they could not explain to students in the first lesson on a new subject why they were about to study it, then it had to be deleted from the curriculum. Sadly that project team could not find a suitable justification for teaching about conics, such as the parabola and ellipse, and so axed them from the course! In the examples above we have described one major reason for investigating parabolas and quadratics. Developments in optics, and the significance of the invention of the reflecting telescope, right up to the current interest in the Hubble telescope, are matters of considerable interest not only in science, but also in the history of ideas and now in the reality of mass worldwide communications. The contribution of people such as Galileo, Kepler and Newton to our understanding of cosmology and gravity are important aspects of a general education. That their work led to mathematical models such as the elliptic orbits of the planets around the Sun and the parabolic trajectory of a bullet from a gun are also important aspects of education.

The discovery of the laws of perspective in renaissance art and architecture is another source of productive links between mathematics, art and history. There are good materials on which to base such work, such as those produced for the Royal Institution's masterclasses by Sir Christopher Zeeman, (see Appendix 11), and the book by Dr JV Field referenced in Appendix 14.

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Some Selective Sources of Teaching Resources

The Association of Teachers of Mathematics (7 Shaftesbury Street, Derby, DE23 8YB) publish a range of resources including: Polygons - Dot to Dot, Squares; Patterns and Quilts; Tiles and Tiling; Transforming; Using Geoboards; Celtic Knot Posters; Tiling Posters; Symmetry Groups; Tiling Generators; Mathematical Activity Tiles; Quilting Tiles; Tiling Generators.

Tarquin Publications (Stradbroke, Diss, Norfolk IP21 5JP) are a good source of materials for teaching geometry. Examples include: Mathematics in Three Dimensions; 3D Geoshapes and Polydron; Escher, Illusions & Perception; Geometrical Pattern Making; Tilings and Tessellations; Paper Engineering and Popups; DIME 3-D Visualising and Thinking; Tangrams, Pentacubes and Pentominoes.

Dale Seymour Publications (P.O.Box 10888, Palo Alto, California, USA) publish some useful materials, including: Mathematical Investigations, Books 1-3 (problem solving tasks covering a range of mathematics, including geometry); Logic Geometry Problems; Blueprint for Geometry (designing and building a scale model of a house); Designing Playgrounds; By Nature's Design (geometry in nature); Structures: The Way Things Are Built; Designing Environments; The Mind's Eye: Imagery in Everyday Life.

A selection of other Royal Society reports and publications

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Acclaim: exploring the lives of leading scientists (112 page curriculum resource pack and 120 minute video for teachers and 11-16 year olds, April 2001, £12.50 ISBN 0 863 39925 8.
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The health hazards of depleted uranium munitions, Part 1 (88 page document 06/01, May 2001, £17.50 ISBN 0 85403 3540; 2 page summary available free of charge.)

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Genetics and Insurance (4-page response to the inquiry by the House of Commons Science and Technology Committee, 03/01, March 2001)

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